1 A Complete Inner Product Space with Dirac's Bracket Notation

A mathematical connection between the bracket notation of quantum mechanics and quaternions is detailed. It will be argued that quaternions have the properties of a complete inner-product space (a Banach space for the field of quaternions). A central issue is the definition of the square of the norm. In quantum mechanics:

$$||\varphi||^2 = <\varphi|\varphi>$$

In this notebook, the following assertion will be examined (* is the conjugate, so the vector flips signs):

$$||(t, \vec{X})||^2 = (t, \vec{X})^* (t, \vec{X}) (t, \vec{X})^* (t, \vec{X})$$

The inner-product of two quaternions is defined here as the transpose (or conjugate) of the first quaternion multiplied by the second. The inner product of a function with itself is the norm.

The Positive Definite Norm of a Quaternion

The square of the norm of a quaternion can only be zero if every element is zero, otherwise it must have a positive value.

$$(t, \vec{X})^* (t, \vec{X}) = (t^2 + \vec{X}.\vec{X}, \vec{0})$$

This is the standard Euclidean norm for a real 4-dimensional vector space.

The Euclidean inner-product of two quaternions can take on any value, as is the case in quantum mechanics for <phi|theta>. The adjective "Euclidean" is used to distinguish this product from the Grassman inner-product which plays a central role in special relativity (see alternative algebra for boosts).

Completeness

With the topology of a Euclidean norm for a real 4-dimensional vector space, quaternions are complete.

Quaternions are complete in a manner required to form a Banach space if there exists a neighborhood of any quaternion x such that there is a set of quaternions y

$$||x - y||^2 < \epsilon^4$$

for some fixed value of epsilon.

Construct such a neighborhood.

$$\left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right)^* \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right) \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right)^* \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right) =$$

$$= \left(\frac{\epsilon^4}{16}, 0, 0, 0\right) < (\epsilon^4, 0, 0, 0)$$

An infinite number of quaternions exist in the neighborhood.

Any polynomial equation with quaternion coefficients has a quaternion solution in x (a proof done by Eilenberg and Niven in 1944, cited in Birkhoff and Mac Lane's "A Survey of Modern Algebra.")

Identities and Inequalities

The following identities and inequalities emanate from the properties of a Euclidean norm. They are worked out for quaternions here in detail to solidify the connection between the machinery of quantum mechanics and quaternions.

The conjugate of the square of the norm equals the square of the norm of the two terms reversed.

$$< \phi | \varphi >^* = < \varphi | \phi >$$

For quaternions,

$$\left(\left(t, \vec{X}\right)^* \left(t', \vec{X'}\right)\right)^* = \left(t\, t' + \vec{X}.\vec{X'}, -t\vec{X'} + \vec{X}t' + \vec{X}x\vec{X'}\right)$$

$$\left(t', \vec{X'}\right)^* \left(t, \vec{X}\right) = \left(t' t + \vec{X'}.\vec{X}, t'\vec{X} - \vec{X'}\, t - \vec{X'}x\vec{X}\right)$$

These are identical, because the terms involving the cross produce will flip signs when their order changes.

For products of squares of norms in quantum mechanics,

$$< \varphi\phi | \varphi\phi > = < \varphi | \varphi >< \phi | \phi >$$

This is also the case for quaternions.

$$< \left(t, \vec{X}\right) \left(t', \vec{X'}\right) \Big| \left(t, \vec{X}\right) \left(t', \vec{X'}\right) > =$$

$$= \left(\left(t, \vec{X}\right) \left(t', \vec{X'}\right)\right)^* \left(t, \vec{X}\right) \left(t', \vec{X'}\right)$$

$$= \left(t', \vec{X'}\right)^* \left(t, \vec{X}\right)^* \left(t, \vec{X}\right) \left(t', \vec{X'}\right)$$

$$= \left(t', \vec{X'}\right)^* (t^2 + x^2 + y^2 + z^2, 0, 0, 0) \left(t', \vec{X'}\right)$$

$$= (t^2 + x^2 + y^2 + z^2, 0, 0, 0) \left(t', \vec{X'}\right)^* \left(t', \vec{X'}\right)$$

$$= \left(t, \vec{X}\right)^* \left(t, \vec{X}\right) \left(t', \vec{X'}\right)^* \left(t', \vec{X'}\right)$$

$$= < \left(t, \vec{X}\right) \Big| \left(t, \vec{X}\right) >< \left(t', \vec{X'}\right) \Big| \left(t', \vec{X'}\right) >$$

The triangle inequality in quantum mechanics:

$$< \varphi + \phi | \phi + \varphi >^2 <= (< \varphi | \varphi > + < \phi | \phi >)^2$$

For quaternions,

$$< \left(t, \vec{X}\right) + \left(t', \vec{X'}\right) \Big| \left(t, \vec{X}\right) + \left(t', \vec{X'}\right) >^2 =$$

$$= \left(\left(t + t', \vec{X} + \vec{X'}\right)^* \left(t + t', \vec{X} + \vec{X'}\right)\right)^2$$

$$= \left(t^2 + t'^2 + \left(\vec{X}\right)^2 + \left(\vec{X'}\right)^2 + 2t\, t' + 2\vec{X}.\vec{X'}, 0\right)^2$$

$$\leq$$

$$\left(t^2 + \left(\vec{X}\right)^2 + t'^2 + \left(\vec{X'}\right)^2 + 2\sqrt{\left(t, \vec{X}\right)^* \left(t, \vec{X}\right) \left(t', \vec{X'}\right)^* \left(t', \vec{X'}\right)}, 0\right) \hat{}2 =$$

$$\left(< \left(t, \vec{X}\right) \Big| \left(t, \vec{X}\right) > + < \left(t', \vec{X'}\right) \Big| \left(t', \vec{X'}\right) >\right)^2$$

If the signs of each pair of component are the same, the two sides will be equal. If the signs are different (a t and a -t for example), then the cross terms will cancel on the left hand side of the inequality, making it smaller than the right hand side where terms never cancel because there are only squared terms.

The Schwarz inequality in quantum mechanics is analogous to dot products and cosines in Euclidean space.

$$|<\varphi|\phi>|^2 <= <\varphi|\varphi><\phi|\phi>$$

Let a third wave function, chi, be the sum of these two with an arbitrary parameter lambda.

$$\chi \equiv \varphi + \lambda \phi$$

The norm of chi will necessarily be greater than zero.

$$(\varphi + \lambda \phi)^* (\varphi + \lambda \phi) = \varphi^* \varphi + \lambda \varphi^* \phi + \lambda^* \phi^* \varphi + \lambda^* \lambda \phi^* \phi >= 0$$

Choose the value for lambda that helps combine all the terms containing lambda.

$$\lambda -> -\frac{\phi^* \varphi}{\phi^* \phi}$$

$$\varphi^* \varphi - \frac{\phi^* \varphi \varphi^* \phi}{\phi^* \phi} >= 0$$

Multiply through by the denominator, separate the two resulting terms and do some minor rearranging.

$$(\varphi^* \phi)^* \varphi^* \phi <= \varphi^* \varphi \ \phi^* \phi$$

This is now the Schwarz inequality.

Another inequality:

$$2 \operatorname{Re} <\varphi|\phi> <= <\varphi|\varphi> + <\phi|\phi>$$

Examine the square of the norm of the difference between two quaternions which is necessarily equal to or greater than zero.

$$0 <= <(t, \vec{X}) - (t', \vec{X'}) | (t, \vec{X}) - (t', \vec{X'})>$$

$$= ((t - t')^2 + (\vec{X} - \vec{X'}).(\vec{X} - \vec{X'}), \vec{0})$$

The cross terms can be put on the other side of inequality, changing the sign, and leaving the sum of two norms behind.

$$(2 (t\ t' + \vec{X}.\vec{X'}), \vec{0}) <= (t^2 + (\vec{X})^2 + t'^2 + (\vec{X'})^2, \vec{0})$$

$$2 \operatorname{Re} <(t, \vec{X}) | (t', \vec{X'})> <= <(t, \vec{X}) | (t, \vec{X})> + <(t', \vec{X'}) | (t', \vec{X'})>$$

The inequality holds.

The parallelogram law:

$$<\varphi + \phi | \phi + \varphi> + <\varphi - \phi | \phi - \varphi> = 2 <\varphi|\varphi> + 2 <\phi|\phi>$$

Test the quaternion norm

$$<(t, \vec{X}) + (t', \vec{X'}) | (t, \vec{X}) + (t', \vec{X'})> + <(t, \vec{X}) - (t', \vec{X'}) | (t, \vec{X}) - (t', \vec{X'})> =$$

$$= ((t + t')^2 + (\vec{X} + \vec{X'}).(\vec{X} + \vec{X'}), \vec{0}) + ((t - t')^2 + (\vec{X} - \vec{X'}).(\vec{X} - \vec{X'}), \vec{0}) =$$

$$= 2(t^2 + (\vec{X})^2 + t'^2 + (\vec{X'})^2, \vec{0}) =$$

$$= 2 <(t, \vec{X}) | (t, \vec{X})> + 2 <(t', \vec{X'}) | (t', \vec{X'})>$$

This is twice the square of the norms of the two separate components.

Implications

In the case for special relativity, it was noticed that by simply squaring a quaternion, the resulting first term was the Lorentz invariant interval. From that solitary observation, the power of a mathematical field was harnessed to solve a wide range of problems in special relativity.

In a similar fashion, it is hoped that because the product of a transpose of a quaternion with a quaternion has the properties of a complete inner product space, the power of the mathematical field of quaternions can be used to solve a wide range of problems in quantum mechanics. This is an important area for further research.

Note: this goal is different from the one Stephen Adler sets out in ”Quaternionic Quantum Mechanics and Quantum Fields.” He tries to substitute quaternions in the place of complex numbers in the standard Hilbert space formulation of quantum mechanics. The analytical properties of quaternions do not play a critical role. It is the properties of the Hilbert space over the field of quaternions that is harnessed to solve problems. It is my opinion that since the product of a transpose of a quaternion with a quaternion already has the properties of a norm in a Hilbert space, there is no need to imbed quaternions again within another Hilbert space. I like a close shave with Occam’s razor.

2 Multiplying Quaternions in Polar Coordinate Form

Any quaternion can be written in polar coordinate form, which involves a scalar magnitude and angle, and a 3-vector I (which in some cases can be the more familiar i).

$$q = ||q|| \, Exp\left[\theta \vec{I}\right] = q^* q \left(Cos[\theta] + \vec{I} \, Sin[\theta]\right)$$

This representation can be useful due to the properties of the exponential function, cosines and sines.

The absolute value of a quaternion is the square root of the norm, which is the transpose of a quaternion multiplied by itself.

$$|q| = \sqrt{q^* q}$$

The angle is the arccosine of the ratio of the first component of a quaternion over the norm.

$$\theta = ArcCos\left(\frac{q + q^*}{2|q|}\right)$$

The vector component is generated by normalizing the pure quaternion (the final three terms) to the norm of the pure quaternion.

$$I = \frac{q - q^*}{2|q - q^*|}$$

I^2 equals -1 just like i^2. Let (0, V) = (q - q*)/2.

$$I^2 = \frac{(0, V)(0, V)}{|(0, V)||(0, V)|} = \frac{(-V.V, VxV)}{(V^2, 0)} = -1$$

It should be possible to do Fourier analysis with quaternions, and to form a Dirac delta function (or distribution). That is a project for the future. Those tools are necessary for solving problems in quantum mechanics.

New method for multiplying quaternion exponentials

Multiplying two exponentials is at the heart of modern analysis, whether one works with Fourier transforms or Lie groups. Given a Lie algebra of a Lie group in a sufficiently small area the identity, the product of two exponentials can be defined using the Campbell-Hausdorff formula:

$$Exp[X] \, Exp[y] = (X + Y) + \frac{1}{2}[X, Y](X + Y)$$

$$+ \frac{1}{12}([[X, Y], Y] - [[X, Y], X])(X + Y) + ...$$

This formula is not easy to use, and is only applicable in a small area around unity. Quaternion analysis that relies on this formula would be very limited.

I have developed (perhaps for the first time) a simpler and general way to express the product of two quaternion exponentials as the sum of two components. The product of two quaternions splits into a commuting and an anti-commuting part. The rules for multiplying commuting quaternions are identical to those for complex numbers. The anticommuting part needs to be purely imaginary. The Grassman product (q q') of two quaternion exponentials and the Euclidean product (q* q') should both have these properties. Together these define the needs for the product of two quaternion exponentials.

$$\text{Let } q = Exp[X] \quad q' = Exp[Y]$$

$$q \, q' = \{q, q'\}^* + Abs[q, q']^* \, Exp\left[\frac{\pi}{2} \frac{[q, q']^*}{Abs[q, q']^*}\right]$$

$$\text{where } \{q, q'\}^* \equiv \frac{q \, q' + q'^* \, q^*}{2} \text{ and } [q, q']^* \equiv q \, q' - q'^* \, q^*$$

$q^* q' = \text{same as above}$

where $\{q, q'\} = q^* q' + q'^* q$ and $[q, q'] = q^* q' - q'^* q$

I call these operators ”conjugators” because they involve taking the conjugate of the two elements. Andrew Millard made the suggestion for the Grassman product that unifies these approaches nicely. What is happening here is that both commuting and anticommuting parts scale themselves appropriately. By using an exponential that has pi/2 multiplied by a normalized quaternion, this always has a zero scalar, as it must to accurately represent an anticommuting part.

3 Commutators and the Uncertainty Principle

Commutators and the uncertainty principle are central to quantum mechanics. Using quaternions in these roles has already been established by others (Horwitz and Biedenharn, Annals of Physics, 157:432, 1984). The first proof of the uncertainty principle I saw relied solely on the properties of complex numbers, not on physics! In this notebook I will repeat that analysis, showing how commutators and an uncertainty principle arise from the properties of quaternions (or their subfield the complex numbers).

Commutators

Any quaternion can be written in a polar form.

$$q = (s, V) = \sqrt{q^* q}\ \mathrm{Exp}\left[\frac{s}{\sqrt{q^* q}} \frac{V}{\sqrt{V^* V}}\right]$$

This is identical to Euler's formula except that the imaginary unit vector i is replaced by the normalized 3-vector. The two are equivalent if j = k = 0. Any quaternion could be the limit of the sum of an infinite number of other quaternions expressed in a polar form. I hope to show that such a quaternion mathematically behaves like the wave function of quantum mechanics, even if the notation is different.

To simplify things, use a normalized quaternion, so that q* q = 1. Collect the normalized 3-vector together with I = V/(V* V)ˆ.5.

The angle s/(q* q)ˆ.5 is a real number. Any real number can be viewed as the product of two other real numbers. This seemingly irrelevant observation lends much of the flexibility seen in quantum mechanics :-) Here is the rewrite of q.

$$q = \mathrm{Exp}[a\, b\, I]$$

$$\text{where}\quad q^* q = 1,\quad a\, b = \frac{s}{\sqrt{q^* q}},\quad I = \frac{V}{\sqrt{V^* V}}$$

The unit vector I could also be viewed as the product of two quaternions.For classical quantum mechanics, this additional complication is unnecessary. It may be required for relativistic quantum mechanics, so this should be kept in mind.

A point of clarification on notation:the same letter will be used 4 distinct ways.There are operators, A hat, which act on a quaternion wave function by multiplying by a quaternion, capital A. If the operator A hat is an observable, then it generates a real number, (a, 0), which commutes with all quaternions, whatever their form. There is also a variable with respect to a component of a quaternion, a_i, that can be used to form a differential operator.

Define a linear operator A hat that multiplies q by the quaternion A.

$$\hat{A}\, q = A\, q$$

If the operator A hat is an observable, then the quaternion A is a real number, (a, 0). This will commute with any quaternion. This equation is functionally equivalent to an eigenvalue equation, with A hat as an eigenvector of q and (a, 0) as the eigenvalue. However, all of the components of this equation are quaternions, not separate structures such as an operator belonging to a group and a vector. This might make a subtle but significant difference for the mathematical structure of the theory, a point that will not be investigated here.

Define a linear operator B hat that multiplies q by the quaternion B. If B hat is an observable, then this operator can be defined in terms of the scalar variable a.

$$\text{Let}\quad \hat{B} = -I \frac{d}{d\, a}$$

$$\hat{B}\, q = -I \frac{d\, \mathrm{Exp}[a\, b\, I]}{d\, a} = b\, q$$

Operators A and B are linear.

$$\left(\hat{A} + \hat{B}\right) q = \hat{A} q + \hat{B} q = a q + b q = (a + b) q$$

$$\hat{A} (q + q') = \hat{A} q + \hat{A} q' = a q + a' q'$$

Calculate the commutator [A, B], which involves the scalar a and the derivative with respect to a.

$$\left[\hat{A}, \hat{B}\right] q = \left(\hat{A} \hat{B} - \hat{B} \hat{A}\right) q = -a I \frac{d q}{d a} + I \frac{d a q}{d a}$$

$$= -a I \frac{d q}{d a} + a I \frac{d q}{d a} + I q \frac{d a}{d a} = I q$$

The commutator acting on a quaternion is equivalent to multiplying that quaternion by the normalized 3-vector I.

The Uncertainty Principle

Use these operators to construct things that behave like averages (expectation values) and standard deviations.

The scalar a–generated by the observable operator A hat acting on the normalized q–can be calculated using the Euclidean product.

$$q^* (A q) = q^* a q = a q^* q = a$$

It is hard to shuffle quaternions or their operators around. Real scalars commute with any quaternion and are their own conjugates. Operators that generate such scalars can move around. Look at ways to express the expectation value of A.

$$q^* \left(\hat{A} q\right) = q^* a q = a q^* q = a^* q^* q = \left(\hat{A} q\right)^* q = a$$

Define a new operator A' based on A whose expectation value is always zero.

$$\text{Let } A' = A - q^* (A q)$$

$$q^* (A' q) = q^* (A - q^* (A q)) q) = a - a = 0$$

Define the square of the operator in a way designed to link up with the standard deviation.

$$\text{Let } DA'^2 = q^* (A'^2 q) - (q^* (A' q))^2 = q^* (A'^2 q)$$

An identical set of tools can be defined for B.

In the section on bracket notation, the Schwarz inequality for quaternions was shown.

$$\frac{A'^* B' + B'^* A'}{2} <= \left|A'\right| \left|B'\right|$$

The Schwarz inequality applies to quaternions, not quaternion operators. If the operators A' and B' are surrounded on both sides by q and q*, then they will behave like scalars.

The left-hand side of the Schwarz inequality can be rearranged to form a commutator.

$$q^* (A'^* B' + B'^* A') q =$$

$$q^* A'^* B' q + q^* B'^* A' q = q^* a'^* B' q + q^* (-I)^* \frac{d}{d a} A' q =$$

$$= q^* a' B' q - q^* (-I) \frac{d}{d a} A' q = q^* (A' B' - B' A') q = q^* [A', B'] q$$

The right-hand side of the Schwarz inequality can be rearranged to form the square of the standard deviation operators.

$$q^* |A'| |B'| q = q^* A'^{*'} A' B'^{*'} B' q = q^* A'^2 B'^2 q = q^* DA'^2 DB'^2 q$$

Plug both of these back into the Schwarz inequality, stripping the primes and the q's which appear on both sides along the way.

$$\frac{[A, B]}{2} <= DA^2 DB^2$$

This is the uncertainty principle for complementary observable operators.

Connections to Standard Notation

This quaternion exercise can be mapped to the standard notation used in physics

bra : $|\psi> \longrightarrow q$

ket : $<\psi| \longrightarrow q^*$

operator : $A \longrightarrow A$

imaginary : $i \longrightarrow I$

commutator : $[A, B] \longrightarrow [A, B]$

norm : $<\psi|\psi> \longrightarrow q^* q$

expectation of A : $<\psi|A\psi>$ maps to $q^* A q$

A is Hermitian $\longrightarrow (0, \vec{A})$ is anti - Hermitian $q^* ((0, \vec{A}) q) = ((0, -\vec{A}) q)^* q$

The square of the standard deviation : $\delta A^2 =<\psi|A^2 \psi> - <\psi|A\psi>^2 \longrightarrow DA^2$

One subtlety to note is that a quaternion operator is anti-Hermitian only if the scalar is zero. This is probably the case for classical quantum mechanics, but quantum field theory may require full quaternion operators. The proof of the uncertainty principle shown here is independent of this issue. I do not yet understand the consequence of this point.

To get to the position-momentum uncertainty equation, make these specific maps

$A \longrightarrow X$

$B \longrightarrow P = i \hbar \frac{d}{dx}$

$I = [A, B] \longrightarrow i \hbar [X, P]$

$$\frac{[A, B]}{2} = \frac{I}{2} <= DA^2 DB^2 \longrightarrow \frac{[X, P]}{2} = \frac{i \hbar}{2} <= \delta X^2 \delta P^2$$

The product of the squares of the standard deviation for position and momentum in the x-direction has a lower bound equal to half the expectation value of the commutator of those operators. The proof is in the structure of quaternions.

Implications

There are many interpretations of the uncertainty principle. I come away with two strange observations. First, the uncertainty principle is about quaternions of the form q = Exp[a b I]. With this insight, one can see by inspection that a plane wave Exp[((Et - P.X)/hbar I], or wave packets that are superpositions of plane waves, will have four uncertainty relations, one for the scalar Et and another three for the three-part scalar P.X. This perspective should be easy to generalize.

Second, the uncertainty principle and gravity are related to the same mathematical properties. This proof of the uncertainty relation involved the Schwarz inequality. It is fairly straightforward to convert that inequality to the triangle inequality. Finding geodesics with quaternions involves the triangle inequality. If a complete theory of gravity can be built from these geodesics (it hasn't yet been done :-) then the inequalities may open connections where none appeared before.

4 Unifying the Representation of Spin and Angular Momentum

I will show how to represent both integral and half-integral spin within the same quaternion algebraic field. This involves using quaternion automorphisms. First a sketch of why this might work will be provided. Second, small rotations in a plane around two axes will be used to show how the resulting vector points in an opposite way, depending on which involution is used to construct the infinitesimal rotation. Finally, a general identity will be used to look at what happens under exchange of two quaternions in a commutator.

Automorphism, Rotations, and Commutators

Quaternions are formed from the direct product of a scalar and a 3-vector. Rotational operators that act on each of the 3 components of the 3-vector act like integral angular momentum. I will show that a rotation operator that acts differently on two of the three components of the 3-vector acts like half-integral spin. What happens with the scalar is irrelevant to this dimensional counting. The same rotation matrix acting on the same quaternion behaves differently depending directly on what involutions are involved.

Quaternions have 4 degrees of freedom. If we want to represent quaternions with automorphisms, 4 are required: They are the identity automorphism, the conjugate anti-automorphism, the first conjugate anti-automorphism, and the second conjugate anti-automorphism:

$$I : q -> q$$

$$* : q -> q^*$$

$$*^1 : q -> q^{*1}$$

$$*^2 : q -> q^{*2}$$

where

$$q^{*1} \equiv (e_1 \, q \, e_1)^*$$

$$q^{*2} \equiv (e_2 \, q \, e_2)^*$$

e1, e2, e3 are orthogonal basis vectors

The most important automorphism is the identity. Life is stable around small permutations of the identity:-) The conjugate flips the signs of the each component in the 3-vector. These two automorphisms, the identity and the conjugate, treat the 3-vector as a unit. The first and second conjugate flip the signs of all terms but the first and second terms, respectively. Therefore these operators act on only the two of the three components in the 3-vector. By acting on only two of three components, a commutator will behave differently. This small difference in behavior inside a commutator is what creates the ability to represent integral and half-integral spins.

Small Rotations

Small rotations about the origin will now be calculated. These will then be expressed in terms of the four automorphisms discussed above.

I will be following the approach used in J. J. Sakurai's book "Modern Quantum Mechanics", chapter 3, making modifications necessary to accommodate quaternions. First, consider rotations about the origin in the z axis. Define:

$$R_{e_3=0}(\theta) \equiv \left(\cos(\theta)\, e_0, 0, 0, \sin(\theta)\, \frac{e_3}{3}\right)$$

$$\text{if } q = \left(0, a_1 \frac{e_1}{3}, a_2 \frac{e_2}{3}, 0\right)$$

$$R_{e_3=0}(\theta)\, q =$$
$$q' = \left(0,\ (a_1 \cos(\theta) - a_2 \sin(\theta))\, e_0 \frac{e_1}{3},\ (a_2 \cos(\theta) + a_1 \sin(\theta))\, e_0 \frac{e_2}{3},\ 0\right)$$

Two technical points. First, Sakurai considered rotations around any point along the z axis. This analysis is confined to the z axis at the origin, a significant but not unreasonable constraint. Second, these rotations are written with generalized coordinates instead of the very familiar and comfortable x, y, z. This extra effort will be useful when considering how rotations are effected by curved spacetime. This machinery is also necessary to do quaternion analysis (please see that section, it's great :-)

There are similar rotations around the first and second axes at the origin;

$$R_{e_1=0}(\theta) = \left(\cos(\theta)\, e_0,\ \sin(\theta) \frac{e_1}{3},\ 0,\ 0\right)$$
$$R_{e_2=0}(\theta) = \left(\cos(\theta)\, e_0,\ 0,\ \sin(\theta) \frac{e_2}{3},\ 0\right)$$

Consider an infinitesimal rotation for these three rotation operators. To second order in theta,

$$\sin(\theta) = \theta + O(\theta^3),\ \cos(\theta) = \left(1 - \frac{\theta^2}{2}\right) + O(\theta^3)$$

$$R_{e_1=0}(\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ \theta \frac{e_1}{3},\ 0,\ 0\right) + O(\theta^3)$$

$$R_{e_2=0}(\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ \theta \frac{e_2}{3},\ 0\right) + O(\theta^3)$$

$$R_{e_3=0}(\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ 0,\ \theta \frac{e_3}{3}\right) + O(\theta^3)$$

Calculate the commutator of the first two infinitesimal rotation operators to second order in theta:

$$[R_{e_1=0}, R_{e_2=0}] = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ \theta \frac{e_1}{3},\ 0,\ 0\right)\left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ \theta \frac{e_2}{3},\ 0\right) -$$
$$- \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ \theta \frac{e_2}{3},\ 0\right)\left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ \theta \frac{e_1}{3},\ 0,\ 0\right) =$$
$$= \left((1 - \theta^2)\, e_0{}^2,\ \theta \frac{e_0 e_1}{3},\ \theta \frac{e_0 e_2}{3},\ \theta^2 \frac{e_1 e_2}{9}\right) -$$
$$\left((1 - \theta^2)\, e_0{}^2,\ \theta \frac{e_0 e_1}{3},\ \theta \frac{e_0 e_2}{3},\ -\theta^2 \frac{e_1 e_2}{9}\right) =$$
$$= 2\left(0,\ 0,\ 0,\ \theta^2 \frac{e_1 e_2}{9}\right) = 2\left(R_{e_3=0}(\theta^2) - R(0)\right)$$

To second order, the commutator of infinitesimal rotations of rotations about the first two axes equals twice one rotation about the third axis given the squared angle minus a zero rotation about an arbitrary axis (a fancy way to say the identity). Now I want to write this result using anti-automorphic involutions for the small rotation operators.

$$[R^*{}_{e_1=0}, R^*{}_{e_2=0}] = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ -\theta \frac{e_1}{3},\ 0,\ 0\right)\left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ -\theta \frac{e_2}{3},\ 0\right) -$$
$$- \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ -\theta \frac{e_2}{3},\ 0\right)\left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ -\theta \frac{e_1}{3},\ 0,\ 0\right) =$$
$$= \left((1 - \theta^2)\, e_0{}^2,\ -\theta \frac{e_0 e_1}{3},\ -\theta \frac{e_0 e_2}{3},\ \theta^2 \frac{e_1 e_2}{9}\right) -$$
$$\left((1 - \theta^2)\, e_0{}^2,\ -\theta \frac{e_0 e_1}{3},\ -\theta \frac{e_0 e_2}{3},\ -\theta^2 \frac{e_1 e_2}{9}\right) =$$
$$= 2\left(0,\ 0,\ 0,\ \theta^2 \frac{e_1 e_2}{9}\right) = 2\left(R_{e_3=0}(\theta^2) - R(0)\right)$$

Nothing has changed. Repeat this exercise one last time for the first conjugate:

$$[R^{*1}{}_{e_1=0}, R^{*1}{}_{e_2=0}] =$$
$$\left(-\left(1-\frac{\theta^2}{2}\right)e_0, \theta\frac{e_1}{3}, 0, 0\right)\left(-\left(1-\frac{\theta^2}{2}\right)e_0, 0, -\theta\frac{e_2}{3}, 0\right)-$$
$$-\left(-\left(1-\frac{\theta^2}{2}\right)e_0, 0, -\theta\frac{e_2}{3}, 0\right)\left(-\left(1-\frac{\theta^2}{2}\right)e_0, \theta\frac{e_1}{3}, 0, 0\right) =$$
$$= \left((1-\theta^2)\,e_0{}^2, -\theta\frac{e_0e_1}{3}, -\theta\frac{e_0e_2}{3}, \theta^2\frac{e_1e_2}{9}\right)-$$
$$\left((1-\theta^2)\,e_0{}^2, -\theta\frac{e_0e_1}{3}, -\theta\frac{e_0e_2}{3}, -\theta^2\frac{e_1e_2}{9}\right) =$$
$$= 2\left(0, 0, 0, \theta^2\frac{e_1e_2}{9}\right) = -2\,(R_{e_3=0}(\theta^2) - R(0))$$

This points exactly the opposite way,even for an infinitesimal angle!

This is the kernel required to form a unified representation of integral and half integral spin. Imagine adding up a series of these small rotations, say 2 pi of these. No doubt the identity and conjugates will bring you back exactly where you started. The first and second conjugates in the commutator will point in the opposite direction. To get back on course will require another 2 pi, because the minus of a minus will generate a plus.

Automorphic Commutator Identities

This is a very specific example. Is there a general identity behind this work? Here it is:

$$[q, q'] = [q^*, q'^*] = [q^{*1}, q'^{*1}]^{*1} = [q^{*2}, q'^{*2}]^{*2}$$

It is usually a good sign if a proposal gets more subtle by generalization :-) In this case, the negative sign seen on the z axis for the first conjugate commutator is due to the action of an additional first conjugate. For the first conjugate, the first term will have the correct sign after a 2 pi journey, but the scalar, third and forth terms will point the opposite way. A similar, but not identical story applies for the second conjugate.

With the identity, we can see exactly what happens if q changes places with q' with a commutator. Notice, I stopped right at the commutator (not including any additional conjugator). In that case:

$$[q, q'] = -[q', q] = [q^*, q'^*] = -[q'^*, q^*] =$$
$$= \left(0, a_2a_3\frac{e_2e_3}{9} + a_3a_2\frac{e_3e_2}{9},\right.$$
$$\left. a_3a_1\frac{e_3e_1}{9} + a_1a_3\frac{e_1e_3}{9}, a_1a_2\frac{e_1e_2}{9} + a_2a_1\frac{e_2e_1}{9}\right)$$
$$[q^{*1}, q'^{*1}] = -[q'^{*1}, q^{*1}] =$$
$$= \left(0, a_2a_3\frac{e_2e_3}{9} + a_3a_2\frac{e_3e_2}{9},\right.$$
$$\left. -a_3a_1\frac{e_3e_1}{9} - a_1a_3\frac{e_1e_3}{9}, -a_1a_2\frac{e_1e_2}{9} - a_2a_1\frac{e_2e_1}{9}\right)$$
$$[q^{*2}, q'^{*2}] = -[q'^{*2}, q^{*2}] =$$
$$= \left(0, -a_2a_3\frac{e_2e_3}{9} - a_3a_2\frac{e_3e_2}{9},\right.$$
$$\left. a_3a_1\frac{e_3e_1}{9} + a_1a_3\frac{e_1e_3}{9}, -a_1a_2\frac{e_1e_2}{9} - a_2a_1\frac{e_2e_1}{9}\right)$$

Under an exchange, the identity and conjugate commutators form a distinct group from the commutators formed with the first and second conjugates. The behavior in a commutator under exchange of the identity automorphism and the anti-automorphic conjugate are identical. The first and second conjugates are similar, but not identical.

There are also corresponding identities for the anti-commutator:

$$\{q, q'\} = \{q^*, q'^*\}^* = -\{q^{*1}, q'^{*1}\}^{*1} = -\{q^{*2}, q'^{*2}\}^{*2}$$

At this point, I don't know how to use them, but again, the identity and first conjugates appear to behave differently that the first and second conjugates.

Implications

This is not a super-symmetric proposal. For that work, there is a super-partner particle for every currently detected particle. At this time, not one of those particles has been detected, a serious omission.

Three different operators had to be blended together to perform this feat: commutators, conjugates and rotations. These involve issue of even/oddness, mirrors, and rotations. In a commutator under exchange of two quaternions, the identity and the conjugate behave in a united way, while the first and second conjugates form a similar, but not identical set. Because this is a general quaternion identity of automorphisms, this should be very widely applicable.

5 Deriving A Quaternion Analog to the Schrödinger Equation

The Schrödinger equation gives the kinetic energy plus the potential (a sum also known as the Hamiltonian H) of the wave function psi, which contains all the dynamical information about a system. Psi is a scalar function with complex values.

$$H\psi = -i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V(0, X)\psi$$

For the time-independent case, energy is written at the operator -i hbar d/dt, and kinetic energy as the square of the momentum operator, i hbar Del, over 2m. Given the potential V(0, X) and suitable boundary conditions, solving this differential equation generates a wave function psi which contains all the properties of the system.

In this section, the quaternion analog to the Schrödinger equation will be derived from first principles. What is interesting are the constraint that are required for the quaternion analog. For example, there is a factor which might serve to damp runaway terms.

The Quaternion Wave Function

The derivation starts from a curious place :-) Write out classical angular momentum with quaternions.

$$\left(0, \vec{L}\right) = \left(0, \vec{R}x\vec{P}\right) = odd\left(\left(0, \vec{R}\right)\left(0, \vec{P}\right)\right)$$

What makes this "classical" are the zeroes in the scalars. Make these into complete quaternions by bringing in time to go along with the space 3-vector R, and E with the 3-vector P.

$$\left(t, \vec{R}\right)\left(e, \vec{P}\right) = \left(Et - \vec{R}.\vec{P}, e\vec{R} + \vec{P}t + \vec{R}x\vec{P}\right)$$

Define a dimensionless quaternion psi that is this product over h bar.

$$\psi \equiv \frac{\left(t, \vec{R}\right)\left(e, \vec{P}\right)}{\hbar} = \left(Et - \vec{R}.\vec{P}, e\vec{R} + \vec{P}t + \vec{R}x\vec{P}\right)/\hbar$$

The scalar part of psi is also seen in plane wave solutions of quantum mechanics. The complicated 3-vector is a new animal, but notice it is composed of all the parts seen in the scalar, just different permutations that evaluate to 3-vectors. One might argue that for completeness, all combinations of E, t, R and P should be involved in psi, as is the case here.

Any quaternion can be expressed in polar form:

$$q = \left|q\right| e^{\arccos\left(\frac{s}{|q|}\right)\frac{\vec{V}}{|\vec{V}|}}$$

Express psi in polar form. To make things simpler, assume that psi is normalized, so |psi| = 1. The 3-vector of psi is quite complicated, so define one symbol to capture it:

$$I \equiv \frac{e\vec{R} + \vec{P}t + \vec{R}x\vec{P}}{\left|e\vec{R} + \vec{P}t + \vec{R}x\vec{P}\right|}$$

Now rewrite psi in polar form with these simplifications:

$$\psi = e^{\left(Et - \vec{R}.\vec{P}\right)I/\hbar}$$

This is what I call the quaternion wave function. Unlike previous work with quaternionic quantum mechanics (see S. Adler's book "Quaternionic Quantum Mechanics"), I see no need to define a vector space with right-hand operator multiplication. As was shown in the section on bracket notation, the Euclidean product of psi (psi* psi) will have all the properties required to form a Hilbert space. The advantage of keeping both operators and the wave function as quaternions is that it will make sense to form an interacting field directly using a product such as psi psi'. That will not be done here. Another advantage is that all the equations will necessarily be invertible.

Changes in the Quaternion Wave Function

We cannot derive the Schrödinger equation per se, since that involves Hermitian operators that acting on a complex vector space. Instead, the operators here will be anti-Hermitian quaternions acting on quaternions. Still it will look very similar, down to the last h bar :-) All that needs to be done is to study how the quaternion wave function psi changes. Make the following assumptions.

1. Energy and Momentum are conserved.

$$\frac{\partial e}{\partial t} = 0 \text{ and} \frac{\partial \vec{P}}{\partial t} = 0$$

2. Energy is evenly distributed in space

$$\vec{\nabla} e = 0$$

3. The system is isolated

$$\vec{\nabla} x \vec{P} = 0$$

4. The position 3-vector X is in the same direction as the momentum 3-vector P

$$\frac{X.P}{|X||P|} = 1 \text{ which implies } \frac{d e^{\vec{I}}}{dt} = 0 \text{ and } \vec{\nabla} x e^{\vec{I}} = 0$$

The implications of this last assumption are not obvious but can be computed directly by taking the appropriate derivative. Here is a verbal explanation. If energy and momentum are conserved, they will not change in time. If the position 3-vector which does change is always in the same direction as the momentum 3-vector, then I will remain constant in time. Since I is in the direction of X, its curl will be zero.

This last constraint may initially appear too confining. Contrast this with the typical classical quantum mechanics. In that case, there is an imaginary factor i which contains no information about the system. It is a mathematical tool tossed in so that the equation has the correct properties. With quaternions, I is determined directly from E, t, P and X. It must be richer in information content. This particular constraint is a reflection of that.

Now take the time derivative of psi.

$$\frac{\partial \psi}{\partial t} = \frac{e\, I}{\hbar} \frac{\psi}{\sqrt{1 + \left(\frac{Et - \vec{R}.\vec{P}}{\hbar}\right)^2}}$$

The denominator must be at least 1, and can be greater that that. It can serve as a damper, a good thing to tame runaway terms. Unfortunately, it also makes solving explicitly for energy impossible unless Et - P.X equals zero. Since the goal is to make a direct connection to the Schrödinger equation, make one final assumption:

5. Et - R.P = 0

$$Et - \vec{R}.\vec{P} = 0$$

There are several important cases when this will be true. In a vacuum, E and P are zero. If this is used to study photons, then t = |R| and E = |P|. If this number happens to be constant in time, then this equation will apply to the wave front.

$$\text{if } \frac{\partial Et - \vec{R}.\vec{P}}{\partial t} = 0,\ e = \frac{\partial \vec{R}}{\partial t}.\vec{P} \text{ or } \frac{\partial \vec{R}}{\partial t} = \frac{e}{\vec{P}}$$

Now with these 5 assumptions in hand, energy can be defined with an operator.

$$\frac{\partial \psi}{\partial t} = \frac{e\, I}{\hbar} \psi$$

$$-I\hbar \frac{\partial \psi}{\partial t} = e\,\psi \text{ or } e = -I\hbar \frac{\partial}{\partial t}$$

The equivalence of the energy E and this operator is called the first quantization.

Take the spatial derivative of psi using the under the same assumptions:

$$\vec{\nabla}\psi = -\frac{\vec{P}\,I}{\hbar}\frac{\psi}{\sqrt{1+\left(\frac{Et-\vec{R}.\vec{P}}{\hbar}\right)^2}}$$

$$\vec{I}\hbar\vec{\nabla}\psi = \vec{P}\,\psi \text{ or } \vec{P} = I\hbar\vec{\nabla}$$

Square this operator.

$$\left(\vec{P}\right)^2 = (mv)^2 = 2m\,\frac{mv^2}{2} = 2m\,KE = -\hbar^2\left(\vec{\nabla}\right)^2$$

The Hamiltonian equals the kinetic energy plus the potential energy.

$$\vec{H}\,\psi = -\vec{I}\,\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\left(\vec{\nabla}\right)^2\psi + V\,\psi$$

Typographically, this looks very similar to the Schrödinger equation. Capital I is a normalized 3-vector, and a very complicated one at that if you review the assumptions that got us here. phi is not a vector, but is a quaternion. This give the equation more, not less, analytical power. With all of the constraints in place, I expect that this equation will behave exactly like the Schrodinger equation. As the constraints are removed, this proposal becomes richer. There is a damper to quench runaway terms. The 3-vector I becomes quite the nightmare to deal with, but it should be possible, given we are dealing with a topological algebraic field.

Implications

Any attempt to shift the meaning of an equation as central to modern physics had first be able to regenerate all of its results. I believe that the quaternion analog to Schrödinger equation under the listed constraints will do the task. These is an immense amount of work needed to see as the constraints are relaxed, whether the quaternion differential equations will behave better. My sense at this time is that first quaternion analysis as discussed earlier must be made as mathematically solid as complex analysis. At that point, it will be worth pushing the envelope with this quaternion equation. If it stands on a foundation as robust as complex analysis, the profound problems seen in quantum field theory stand a chance of fading away into the background.

6 Introduction to Relativistic Quantum Mechanics

The relativistic quantum mechanic equation for a free particle is the Klein-Gordon equation (h=c=1)

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\Psi = 0$$

The Schrödinger equation results from the non-relativistic limit of this equation. In this section, the machinery of the Klein-Gordon equation will be ported to quaternions.

The Wave Function

The wave function is the superposition of all possible states of a system. The product of the conjugate of a wave function with another wave function forms a complete inner product space. In the energy/momentum representation, this would involve all possible energy levels and momenta.

$$\Psi \equiv \textbf{the sum from } n = 0 \textbf{ to infinity of } \left(e_n, \left(\vec{P}\right)_n\right)$$

This infinite sum of quaternions should contain all the information about a system. The quaternion wave function can be normalized.

$$\sum_{n=0}^{\infty}\left(e_n, \left(\vec{P}\right)_n\right)^*\left(e_n, \left(\vec{P}\right)_n\right) = \sum_{n=0}^{\infty}\left(e_n^2 + \vec{P}_n^2, 0\right) = 1$$

The first quaternion is the conjugate or transpose of the second. Since the transpose of a quaternion wave function times a wave function creates a Euclidean norm, this representation of wave functions as an infinite sum of quaternions can form a complete, normed product space.

The Klein-Gordon Equation

The Klein-Gordon equation can be divided into two operators that act on the wave function: the D'Alembertian and the scalar mˆ2. The quaternion operator required to create the D'Alembertian, along with vector identities, has already been worked out for the Maxwell equations in the Lorenz gauge.

$$\sum_{n=0}^{\infty}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^2 + \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)^2\right)\left(e_n, \left(\vec{P}\right)_n\right)\Big/2 =$$

$$= \sum_{n=0}^{\infty}\left(\frac{\partial^2 e_n}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}e_n - \vec{\nabla}\cdot\vec{\nabla}X\left(\vec{P}\right)_n, \frac{\partial^2\left(\vec{P}\right)_n}{\partial t^2} - \vec{\nabla}\vec{\nabla}\cdot\left(\vec{P}\right)_n + \vec{\nabla}X\vec{\nabla}X\left(\vec{P}\right)_n + \vec{\nabla}X\vec{\nabla}e_n\right)$$

The first term of the scalar, and the second term of the vector, are both equal to zero. What is left is the D'Alembertian operator acting on the quaternion wave function.

To generate the scalar multiplier mˆ2, substitute En and Pn for the operators d/dt and del respectively, and repeat. Since the structure of the operator is identical to the previous one, instead of the D'Alembertian times the wave function, there is Enˆ2-Pnˆ2. The sum of all these terms becomes mˆ2.

Set the sum of these two operators equal to zero to form the Klein-Gordon equation.

$$\sum_{n=0}^{\infty}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^2 + \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)^2 + \left(e_n, \left(\vec{P}\right)_n\right)^2 + \left(e_n, -\left(\vec{P}\right)_n\right)^2\right)\left(e_n, \left(\vec{P}\right)_n\right)\Big/2 =$$

$$= \sum_{n=0}^{\infty}\left(-\vec{\nabla}\cdot\left(\vec{\nabla}X\left(\vec{P}\right)_n\right) - \vec{\nabla}\cdot\vec{\nabla}e_n - \left(\vec{P}\right)_n\cdot\left(\left(\vec{P}\right)_n X\left(\vec{P}\right)_n\right) - \left(\left(\vec{P}\right)_n\cdot\left(\vec{P}\right)_n\right)e_n + e_n{}^3 + \frac{\partial^2 e_n}{\partial t^2},\right.$$

$$\vec{\nabla}\text{X}\left(\vec{\nabla}\text{X}\left(\vec{P}\right)_n\right) + \vec{\nabla}\text{X}\left(\vec{\nabla}e_n\right) + \left(\vec{P}\right)_n\text{X}\left(\left(\vec{P}\right)_n\text{X}\left(\vec{P}\right)_n\right) + \left(\left(\vec{P}\right)_n\text{X}\left(\vec{P}\right)_n\right)e_n - \vec{\nabla}$$

$$\left(\vec{\nabla}\cdot\left(\vec{P}\right)_n\right) + \left(\vec{P}\right)_n e_n^2 - \left(\vec{P}\right)_n\left(\left(\vec{P}\right)_n\cdot\left(\vec{P}\right)_n\right) + \frac{\partial^2\left(\vec{P}\right)_n}{\partial t^2}\Bigg)$$

It takes some skilled staring to assure that this equation contains the Klein-Gordon equation along with vector identities.

Connection to the Maxwell Equations

If m=0, the quaternion operators of the Klein-Gordon equation simplifies to the operators used to generate the Maxwell equations in the Lorenz gauge. In the homogeneous case, the same operator acting on two different quaternions equals the same result. This implies that

$$\left(\varphi,\ \vec{A}\right) = \sum_{n=0}^{\infty}\left(e_n,\ \left(\vec{P}\right)_n\right)$$

Under this interpretation, a nonzero mass changes the wave equation into a simple harmonic oscillator. The simple relationship between the quaternion potential and the wave function may hold for the nonhomogeneous case as well.

Implications

The Klein-Gordon equation is customarily viewed as a scalar equation (due to the scalar D'Alembertian operator) and the Maxwell equations are a vector equation (due to the potential four vector). In this notebook, the quaternion operator that generated the Maxwell equations was used to generate the Klein-Gordon equation. This also created several vector identities which are usually not mentioned in this context. A quaternion differential equation is needed to perform the work of the Dirac equation, but since quaternion operators are a field, an operator that does the task must exist.

7 Time Reversal Transformations for Intervals

The following transformation R for quaternions reverses time:

$$\left(t, \vec{X}\right) -> \left(-t, \vec{X}\right) = R\left(t, \vec{X}\right)$$

The quaternion R exist because quaternions are a field.

R will equal (-t, X)(t, $X)^{-1}$. The inverse of quaternion is the transpose over the square of the norm, which is the scalar term of the transpose of a quaternion times itself.

$$R = \left(-t, \vec{X}\right)\left(t, \vec{X}\right)^{-1} = \left(-t^2 + \vec{X}.\vec{X}, 2\,t\,\vec{X}\right)\Big/\left(t^2 + \vec{X}.\vec{X}\right)$$

For any given time, R can be defined based on the above.

Classical Time Reversal

Examine the form of the quaternion which reverses time under two conditions. A interval normalized to the interval takes the form (1, beta), a scalar one and a 3-vector relativistic velocity beta . In the classical region, beta<<<1. Calculate R in this limit to one order of magnitude in beta.

$$R = \left(-t, \vec{\beta}\right)\left(t, \vec{\beta}\right)^{-1} = \left(-t^2 + \vec{\beta}.\vec{\beta}, 2\,t\,\vec{\beta}\right)\Big/\left(t^2 + \vec{\beta}.\vec{\beta}, 0\right)$$

$$\text{if } \beta << 1 \text{ then } R \approx \left(-1, 2\,t\,\vec{\beta}\right)$$

The operator R is almost the negative identity, but the vector is non-zero, so it would not commute.

Relativistic Time Reversal

For a relativistic interval involving one axis, the interval could be characterized by the following:

$$(T + \epsilon, T, 0, 0)$$

Find out what quaternion is required to reverse time for this relativistic interval to first order in epsilon.

$$R = \left(\frac{T^2 - (T+\epsilon)^2}{T^2 + (T+\epsilon)^2}, \frac{2\,T\,(T+\epsilon)}{T^2 + (T+\epsilon)^2}, 0, 0\right) = \left(-\frac{\epsilon}{T} + O[\epsilon]^2, 1 + O[\epsilon]^2, 0, 0\right)$$

This approaches q[-e/T, 1, 0, 0], almost a pure vector, a result distinct from the classical case.

Implications

In special relativity, the interval between events is considered to be 4 vector are operated on by elements of the Lorentz group. The element of this group that reverses time has along its diagonal

{-1, 1, 1, 1}, zeroes elsewhere. There is no dependence on relative velocity. Therefore special relativity predicts the operation of time reversal should be indistinguishable for classical and relativistic intervals. Yet classically, time reversal appears to involve entropy, and relativistically, time reversal involves antiparticles.

In this notebook, a time reversal quaternion has been derived and shown to work. Time reversal for classical and relativistic intervals have distinct limits, but these transformations have not yet been tied explicitly to the laws of physics.