Dynamic Graphs, Quaternion Analysis, and Unified Field Theory

Doug Sweetser

Email: sweetser@alum.mit.edu

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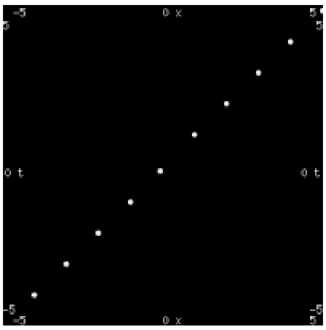

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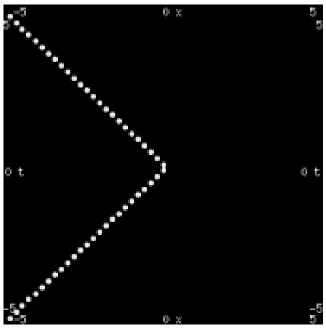

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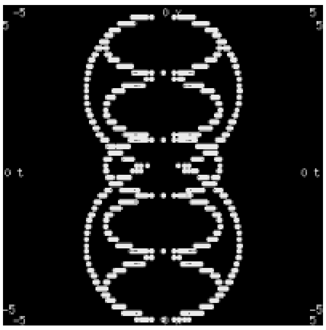

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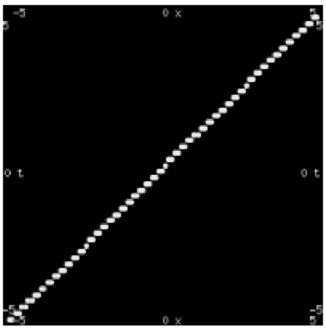

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1.1 Mathematical Physics

The study of local change in patterns of spacetime events.

- Local, near here and now, no global answers to all questions.
- Change = calculus.
- Spacetime: 4-vectors (+, * scalar) with the structure of a field (+, -, *, /).
- Events: this happened here then.

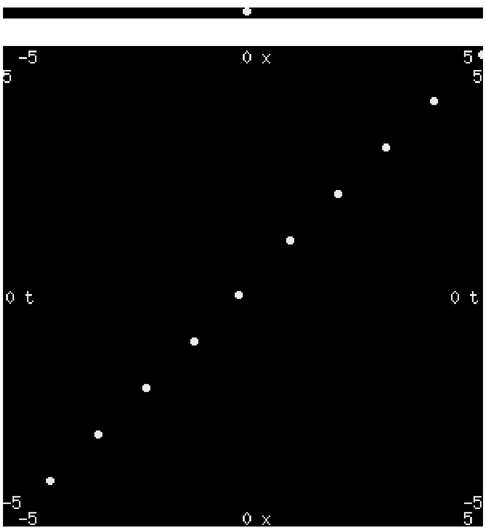
Events are the fundamental currency of the Universe. Event can be huge like the big bang, or tiny, like an electron continuing to flutter around an atom.

Use physics to motivate math.

1.2 Dynamics Graphs of Complex Numbers

Real t = time of an event.

Imaginary x = a position in space.



10 events, where t = x.

- An event is discrete - no half events.
- Number of events is equal in the two graphs.

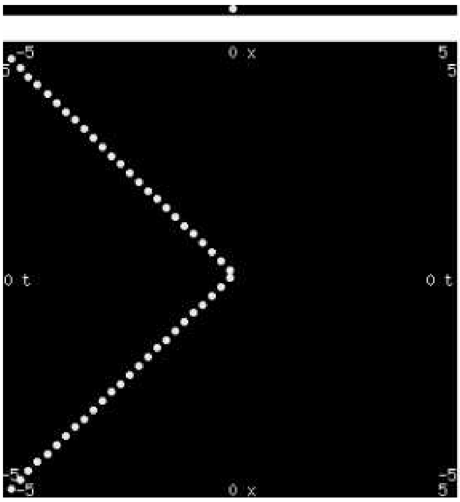
- Static graph below is generated by snapshots of the [dynamic] graph above.

Note: go to http://sdm.openacs.org/wp/display/1238 for a demonstration of dynamic graphs.

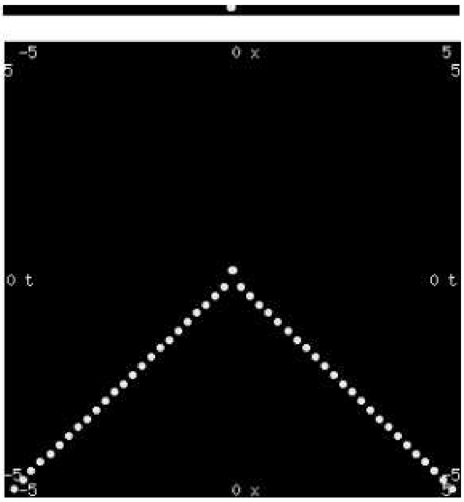
1.3 Origin is Unique

Additive inverse $(0, 0)$ is special. Everything with a worldline defines its own origin, the place for that object to collect information about the rest of the Universe.

- now = center of events with time reflections.
- here = center of events with space reflections.



Time reflection of 5 events.



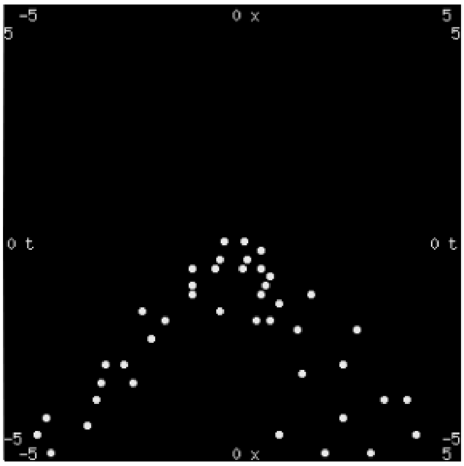
Space reflection of 5 events.

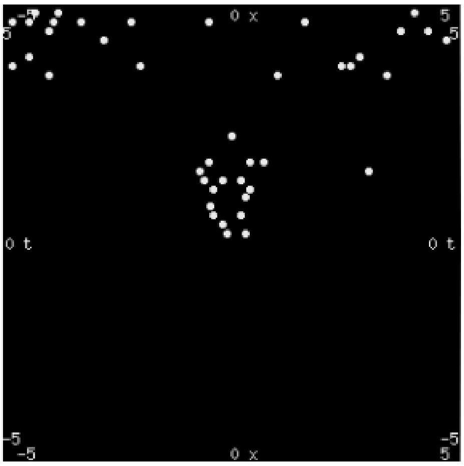
A time mirror is distinct from a space mirror for a dynamic representation of complex numbers.

- Time reflection: events go out the way they came in.
- Space reflection: pairs of equidistant events.

1.4 Around the Origin

- Before: Happened recently, near here.
- After: Will happen soon, near here.



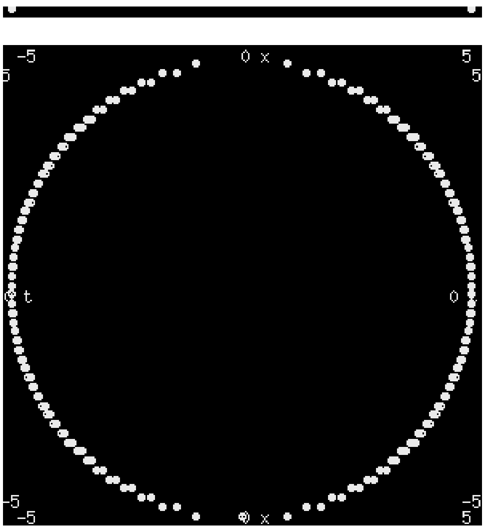


Up to now. The future.

Data up to now can be used to estimate the future.

1.5 Big Circles

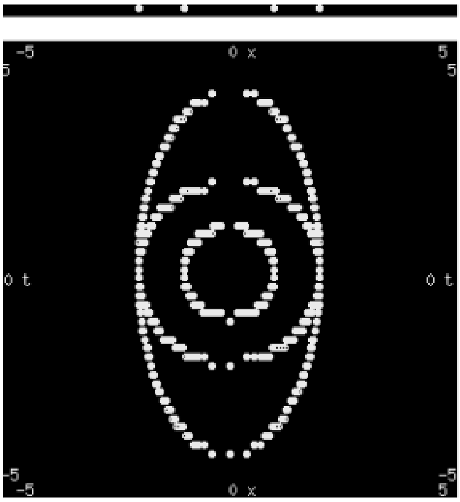
Here is a dynamic graph of a complex-valued oscillator.



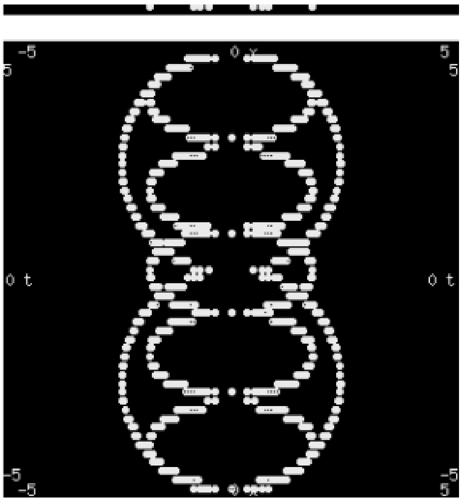
- Near now ($t = 0$), there are many events and movement in space shrinks.
- Near here ($x = 0$), there are few events and large spatial jumps.

1.6 Multiple Frequencies

Oscillation at several frequencies.



Three wavelengths: 2, 4, and 8.

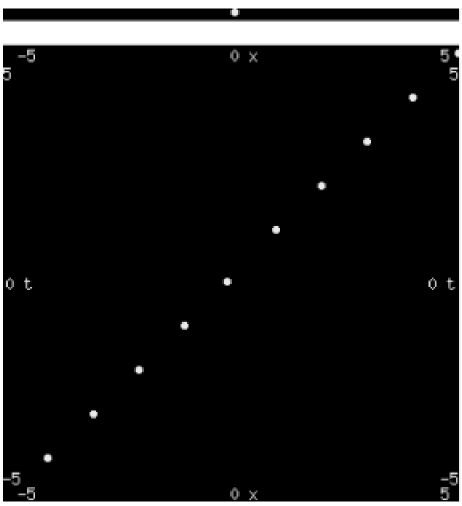


Two wavelengths: 2 and 5.

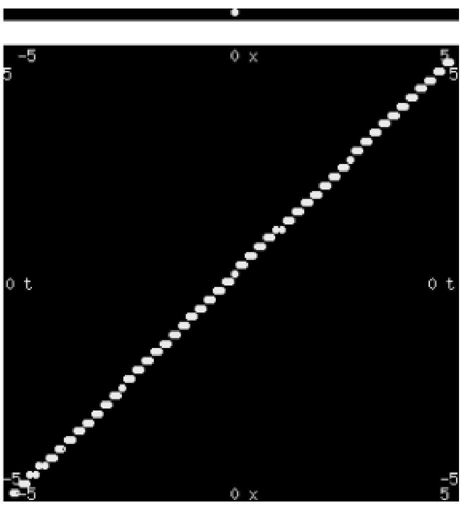
1.7 Seeing Uncertainty

- Time is discrete between frames.
- Time is a continuous, completely ordered set within frames.
- Events cannot be completely ordered.

Same generating function, different information density.



1 event/5 frames



2 events/frame

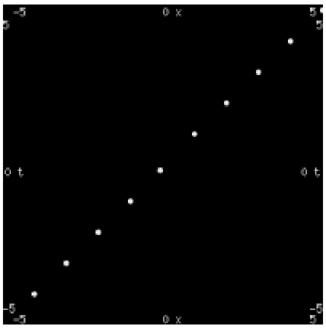
Uncertainty is minimized with low information density.

1.8 Summary: Dynamics Graphs of Complex Numbers

Math:

Treat the real part of complex numbers as a dynamic variable in an animation, the imaginary part as a location in space.

Pictures:



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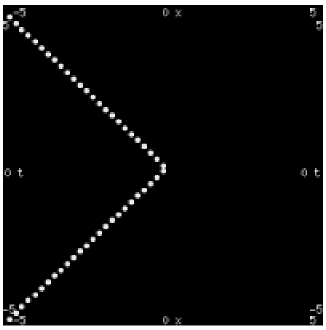

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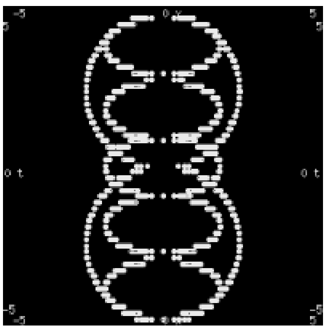

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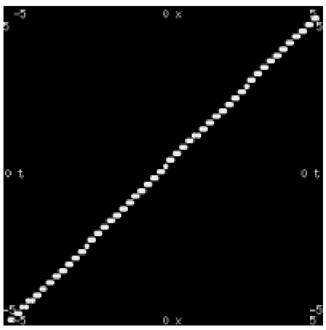

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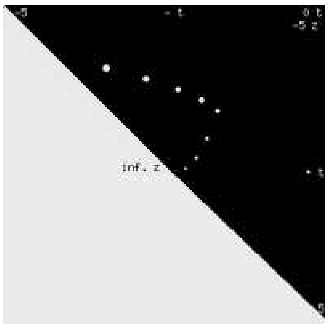
Chapter 2
Dynamic Graphs of Quaternions

1. Dynamic graphs of quaternions.
2. Key for dynamic graphs.
3. Time reflection.
4. Space reflection.
5. Tilted big circles.
6. Circles and reflections.
7. Frequencies and amplitudes.



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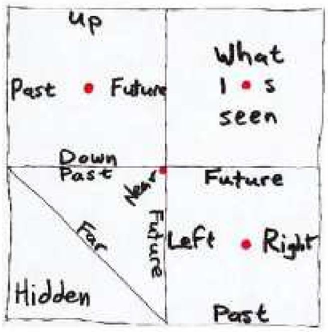

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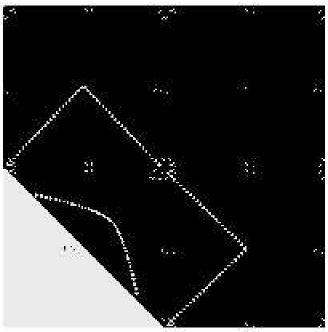

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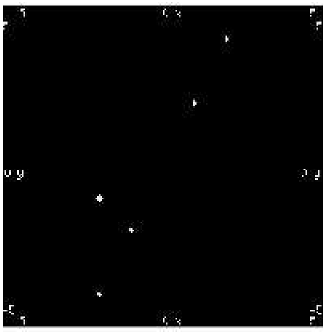

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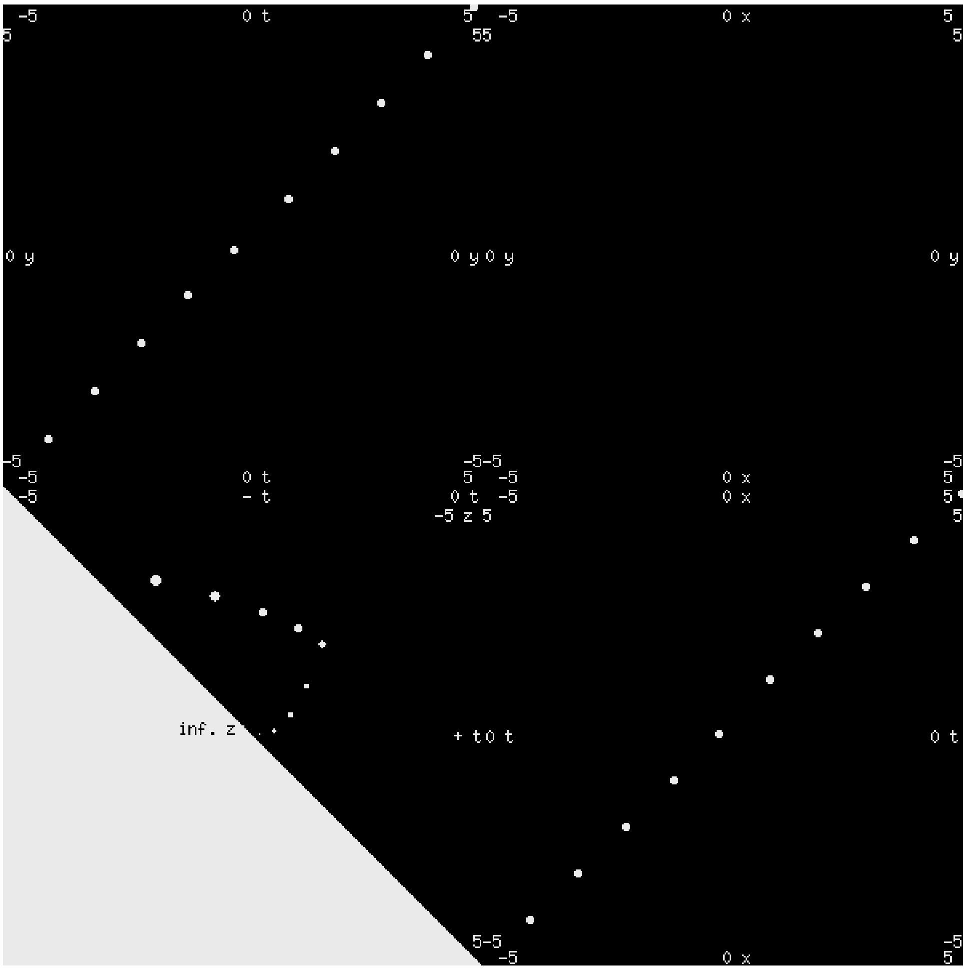
2.1 Dynamics Graphs of Quaternions

Real t = time of an event.

Imaginary x, y, z = positions in space.



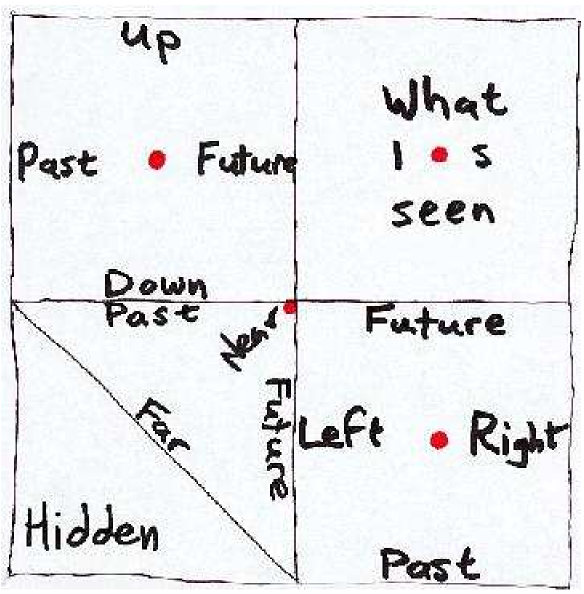
10 events, where $t = x = y = z$.

- Three static complex planes: time with vertical, horizontal, and perspective position in space.
- Perspective plane has no information "behind" viewer.
- Dynamic graph has events that change size.

2.2 Key for Dynamic Graphs

Each event has four types of information:

1. When it happened.
2. How far up it happened.
3. How far left it happened.
4. How far away it happened.

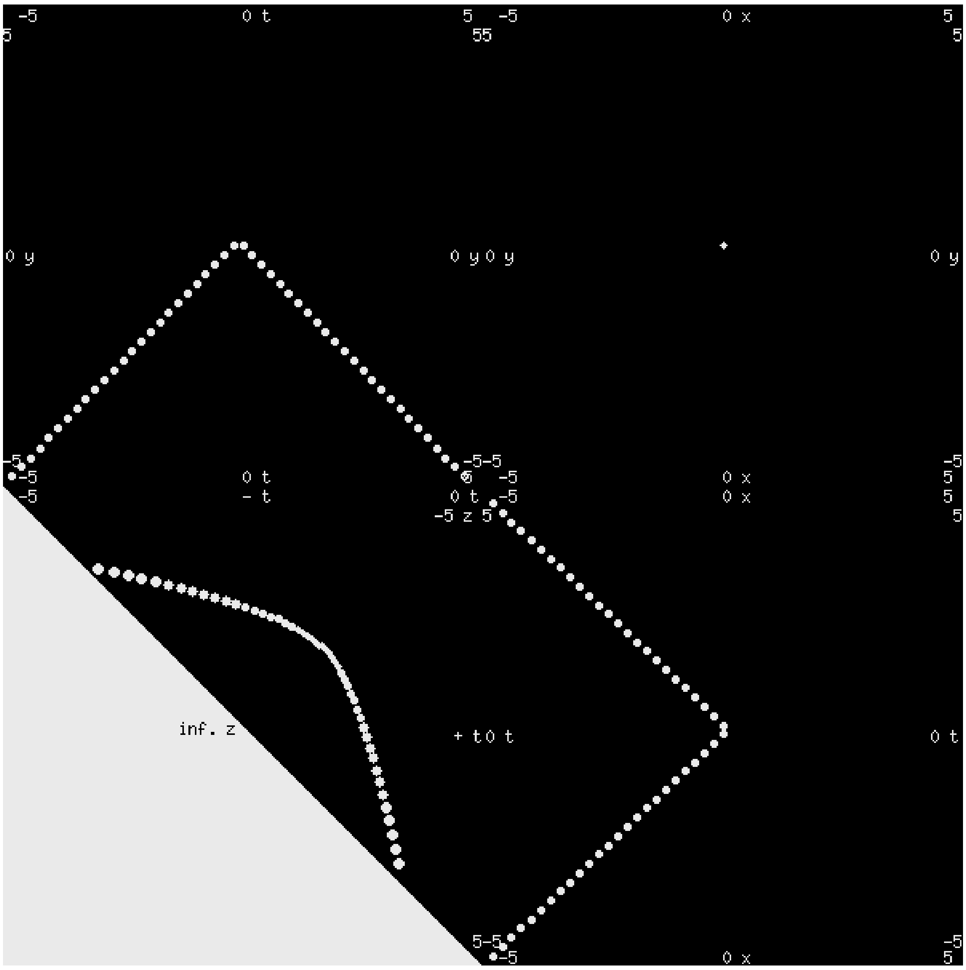


Each of the four panels has time, the unifying thread. One static graph also has how up, another how left, a third how far away the events were/will be. The dynamic graph integrates all that information.

There are eight boundaries: past and future, up and down, left and right, near and infinitely far. All events that appear in the four graphs must be within these eight constraints.

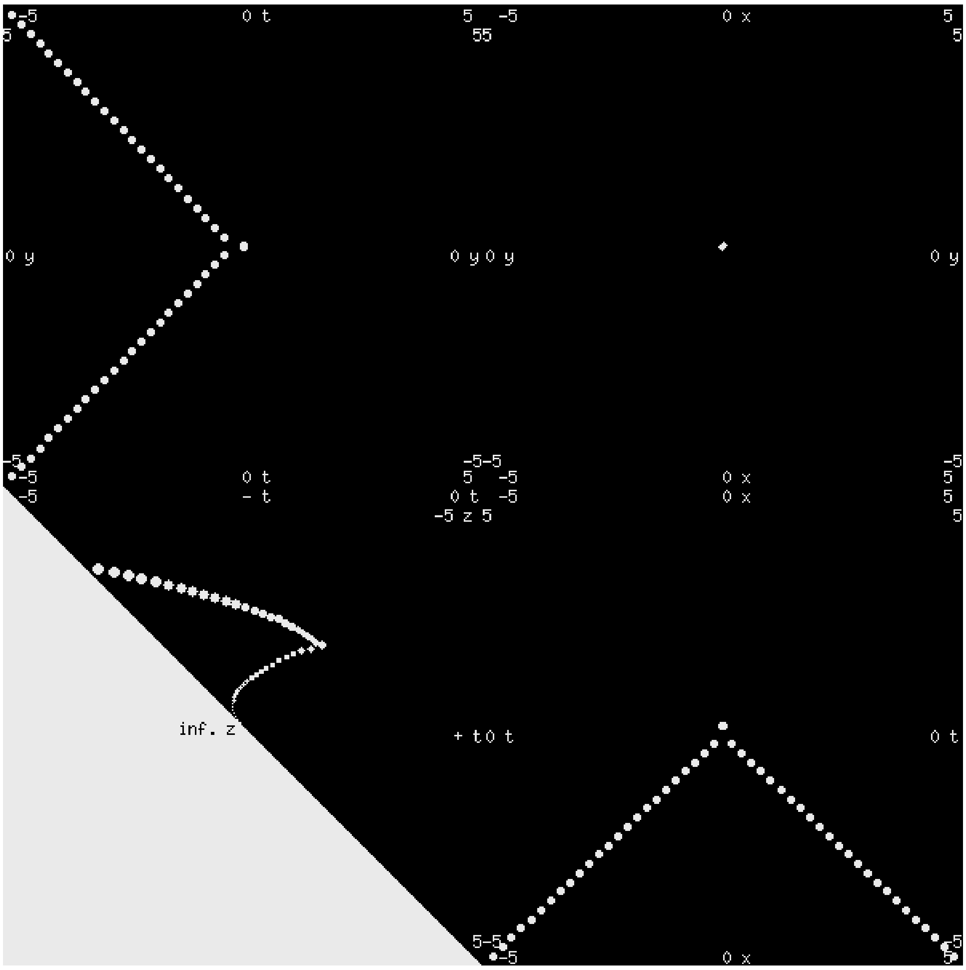
2.3 Time Reflection

Events go out the way they came in for x, y, and z.



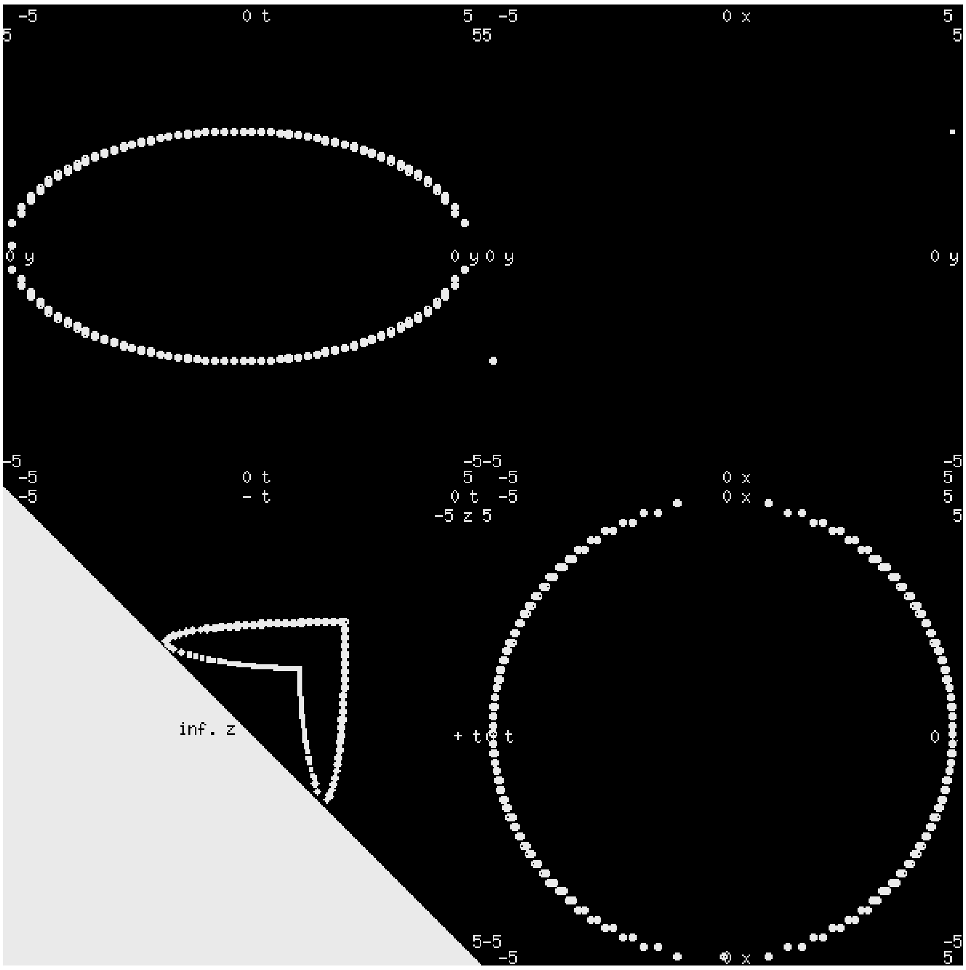
2.4 Space Reflection

Pairs of equidistant events for x, y, and z, except at the origin.



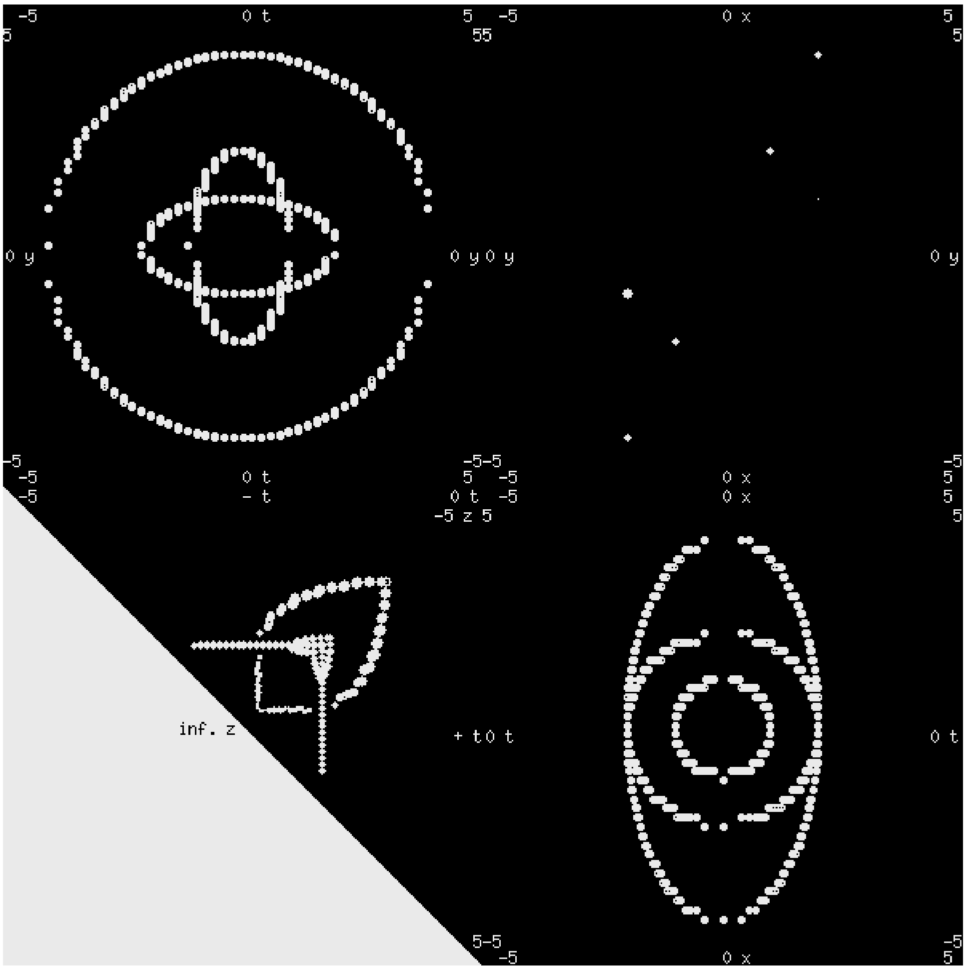
2.5 Tilted Big Circle

The oscillator is tilted away and down from an observer, but not twisted to the left.



2.6 Circles and Reflections

Pairs of equidistant events go out as they came in, so oscillators have both time and space reflections.



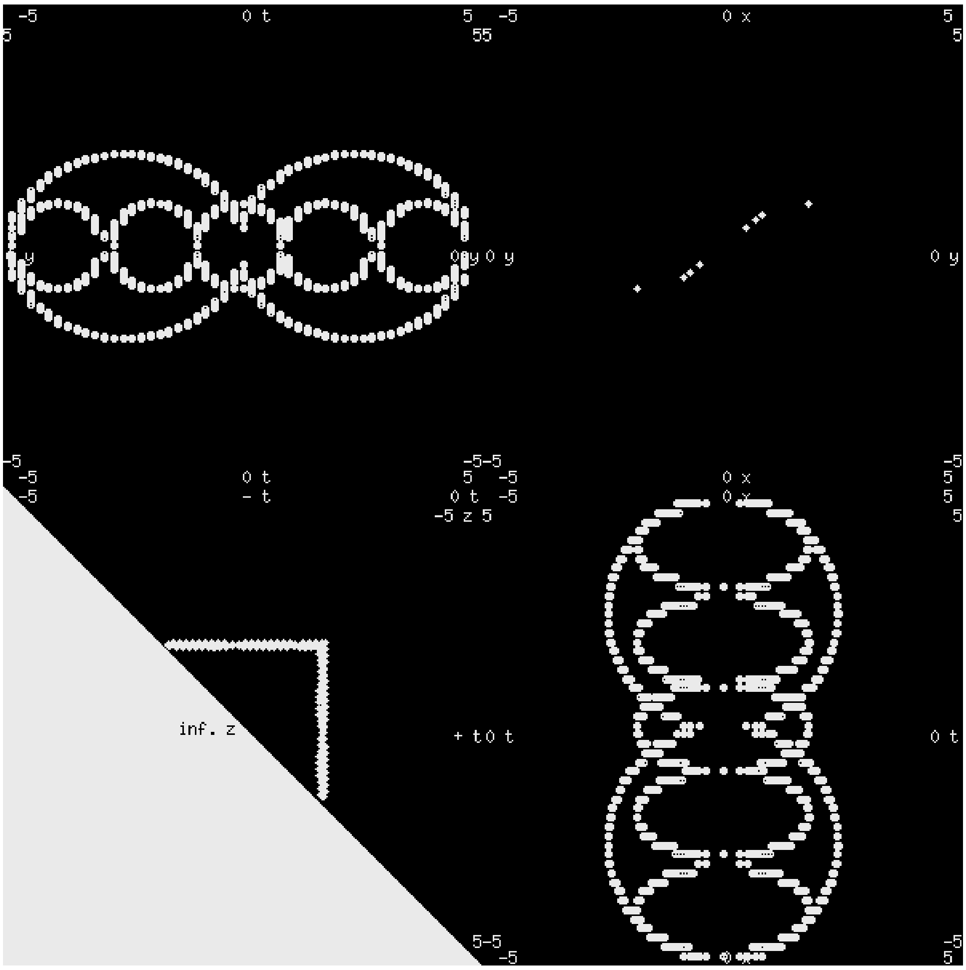
2.7 Frequencies and Amplitudes

Frequency is how often something happens again in a set amount of time, an inverse measure of time. An amplitude is how far in space an oscillator travels.

- The frequencies are shared in all four graphs.

- The amplitudes are independent in the three static graphs.

- The dynamic graph is precisely limited by what happens in the static graphs.

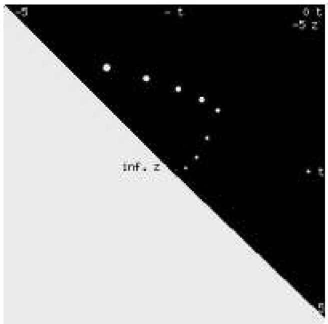


2.8 Summary: Dynamic Graphs of Quaternions

Math:

Three static complex graphs combine to make one dynamic graph of a quaternion.

Pictures:



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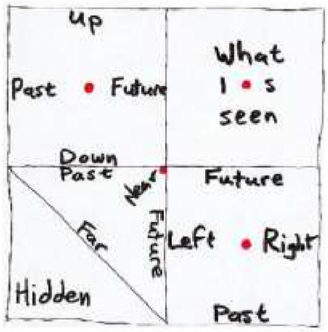

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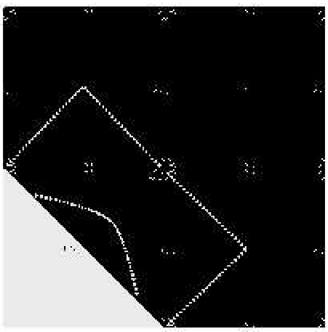

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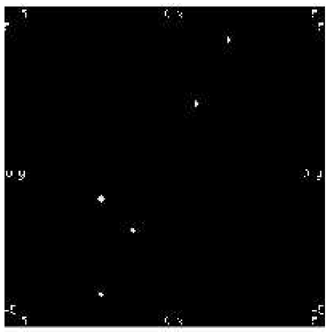

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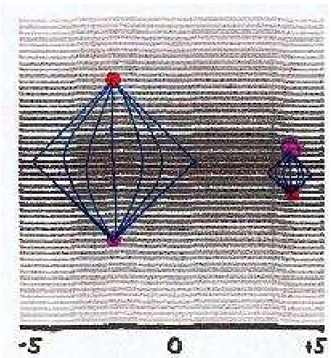
Chapter 3

Quaternions Analysis

3.1 Division: Normalize by All Histories between Conjugates

$n^{-1} = \frac{n^*}{\mathrm{Norm}(n)}$

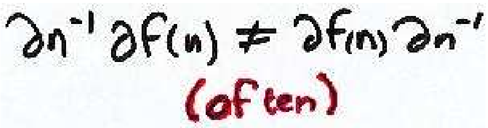
- Works for $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. 3 fields, 1 definition.
- For $\mathbb{R}$, $n^* = n$, $\mathrm{Norm}(n) = n^2$.
  Overly complex, but works without modification for $\mathbb{C}$ and $\mathbb{H}$.
- Conjugate = mirror reflection.
- All histories between conjugates are all timelike paths bounded by lightcones.



3.2 Derivative: Division in a Limit

$\frac{\partial f(n)}{\partial n} = \lim_{dn \to 0} \frac{f(n + dn) - f(n)}{dn}$

- Works for $\mathbb{R}$ and $\mathbb{C}$.
- Fails for $\mathbb{H}$.



The key: define the limit process so that the differential elements always commute.

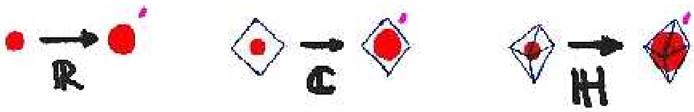
3.3 Better Derivative: Division in a 2-Step Limit Process

Let the 3-vector of the differential element go to zero first, followed by the scalar.

$$\frac{\partial f(n)}{\partial n} = \lim_{(\frac{dn+dn^*}{2} \to 0)} \left(\lim_{(\frac{dn-dn^*}{2} \to 0)} \frac{f(n+dn)-f(n)}{dn} \right)$$

- For $\mathbb{R}$, $\frac{dn+dn^*}{2}=0$, $\frac{dn-dn^*}{2}=dn$.
  Overly complex, but works.
- For $\mathbb{C}$, free to change order of limits.
- For $\mathbb{H}$, not free to change order.

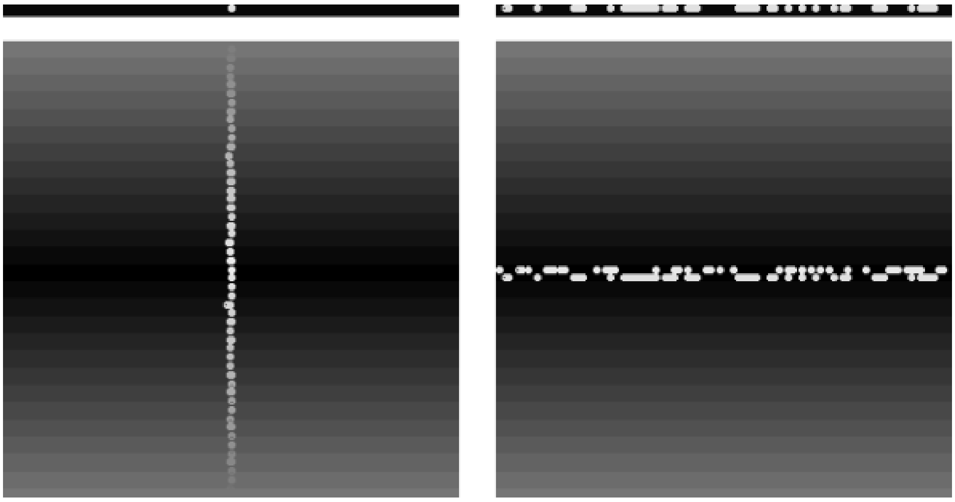
Complex numbers are symmetric for time and 1-space. Quaternion 3-space breaks the symmetry with time, so the order of limits matters.



3.4 Causal Order of Limits: Timelike, Spacelike, or Lightlike

1. Start with the differential element:
   $dn = (dt, d\vec{R}/c)$
2. Rescale to the Lorentz invariant interval $(d\tau)^2 = (dt)^2 - (d\vec{R}/c)^2$:
   $\frac{dn}{d\tau} = (\frac{dt}{d\tau}, \frac{d\vec{R}}{c\,d\tau})$
3. If the rescaled differential element's **3-vector** goes to zero first, $\frac{dn}{d\tau} + \frac{dn^*}{d\tau}/2 \to 0$, the velocity approaches zero.
   The interval between the differential elements is timelike.
   Classical mechanics applies.
4. If the rescaled differential element's **scalar** goes to zero first, $\frac{dn}{d\tau} - \frac{dn^*}{d\tau}/2 \to 0$, the derivative $\frac{df(n)}{dn}$ is **not** defined, but the norm of the derivative, $(\frac{df(n)}{dn})^* \frac{df(n)}{dn}$, **is**.
   The interval between the differential element events is spacelike.

Classical quantum mechanics applies.



Classical: Know much about a little space.
Classical QM: Much about a little time.

3.5 General Balanced Basis Vectors

Do not specify the coordinate system (it could be Cartesian or spherical for example, but it does not matter).

Need to balance scalar with sum of all three imaginaries. Holonomic equation test for an analytic quaternion function requires the balance between the real and imaginary.

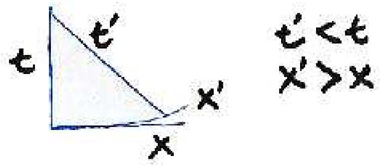
- $\mathbb{H}$: $e_0, \frac{e_1}{3}, \frac{e_2}{3}, \frac{e_3}{3}$.
- $\mathbb{C}$: $e_0, I = \frac{e_1}{3} + \frac{e_2}{3} + \frac{e_3}{3}$.
- $\mathbb{R}$: $e_0, I = 0$.

3.6 Length of Basis Vectors

1. Hamilton's convention:
   $1 = e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = -e_1 e_2 e_3$
2. In the Schwarzschild solution of general relativity:
   $1 = e_0^2 = -e_1^2 = -e_2^2 = -e_3^2$ iff flat.
   $e_0^2 < 1$ in curved spacetime.

$e_{\vec{R}}^2 > 1$ in curved spacetime.

3. Consistent with the above observations, define a new convention for basis vector length:
   $1 - \triangle = e_0^2 = -\frac{1}{e_1^2} = -\frac{1}{e_2^2} = -\frac{1}{e_3^2}$ ($\triangle = 0$ iff flat)



3.7 Span of Automorphisms

The fields ℝ, ℂ, and ℍ have 1, 2, and 4 degrees of freedom. To represent all possible functional mappings, 1, 2, and 4 automorphic functions are required for each field.

- ℝ: Identity (x).
- ℂ: Identity, Conjugate (z, z^*).
- ℍ: Identity, 3 Conjugates (q, q^*, q^{*1}, q^{*2}) where
  $q^{*1} \equiv (e_1 \, q \, e_1)^* = (-t, x, -y, -z)$,
  $q^{*2} \equiv (e_2 \, q \, e_2)^* = (-t, -x, y, -z)$.

Any quaternion function on the $\mathbb{R}^4$ manifold can be represented on the manifold $\mathbb{H}^1$ using a combination of q, q^*, q^{*1}, and q^{*2}.

3.8 Four Analytic Function Tests

1. Apply limit definition and show only one non-zero derivative with respect to q, q^*, q^{*1}, and q^{*2}.
2. The Cauchy-Riemann Equations.
3. The holonomic equation.
4. The chain rule.

3.9 Analytic by Derivative Definition

1. Start with a simple polynomial:

$f = q^2$

2. Apply the definition:
$\frac{\partial f}{\partial q} = \lim_{(d,\vec{0})\to 0} \left(\lim_{(d,\vec{D})\to(d,\vec{0})} \frac{(q+(d,\vec{D}))^2 - q^2}{(d,\vec{D})} \right)$

3. Expand:
$\frac{\partial f}{\partial q} = \lim_{(d,\vec{0})\to 0} \left(\lim_{(d,\vec{D})\to(d,\vec{0})} \frac{q^2 + q(d,\vec{D}) + (d,\vec{D})q + (d,\vec{D})^2 - q^2}{(d,\vec{D})} \right)$

4. Apply limits:
$\frac{\partial f}{\partial q} = \lim_{(d,\vec{0})\to 0} \frac{q(d,\vec{0}) + (d,\vec{0})q + (d,\vec{0})^2}{(d,\vec{0})} = 2\,q$

5. The function f does not depend on q^*, q^{*1}, or q^{*2}:
$\frac{\partial f}{\partial q^*} = \frac{\partial f}{\partial q^{*1}} = \frac{\partial f}{\partial q^{*2}} = 0$

f is analytic in q.

3.10 Analytic by the Cauchy-Riemann Equations

1. Start with the same simple polynomial:
$f = q^2$

2. Write out its components:
$q^2 = (t^2\, e_0^2 + x^2\, \frac{e_1^2}{9} + y^2\, \frac{e_2^2}{9} + z^2\, \frac{e_3^2}{9}, 2\,t\,x\,e_0\,\frac{e_1}{3}\,, 2\,t\,y e_0\,\frac{e_2}{3}\,, 2\,t\,z e_0\,\frac{e_3}{3})$

3. Split into a scalar function u and a 3-vector function $\vec{V}$:
$u = t^2\, e_0^2 + x^2\, \frac{e_1^2}{9} + y^2\, \frac{e_2^2}{9} + z^2\, \frac{e_3^2}{9}$
$\vec{V} = (2\,t\,x\,e_0\,\frac{e_1}{3}\,, 2\,t\,y e_0\,\frac{e_2}{3}\,, 2\,t\,z e_0\,\frac{e_3}{3})$

4. Compare the product of the time derivative of u with the 3-vector I to the product of the spatial derivatives of $\vec{V}$ and the scalar basis vector e_0:
$\frac{\partial u}{\partial t}\,\frac{e_1}{3} = \frac{2}{3}\,t e_0^2\, e_1 \quad \frac{\partial V_x}{\partial x}\, e_0 = \frac{2}{3}\,t e_0^2\, e_1$
$\frac{\partial u}{\partial t}\,\frac{e_2}{3} = \frac{2}{3}\,t e_0^2\, e_2 \quad \frac{\partial V_y}{\partial y}\, e_0 = \frac{2}{3}\,t e_0^2\, e_2$
$\frac{\partial u}{\partial t}\,\frac{e_3}{3} = \frac{2}{3}\,t e_0^2\, e_3 \quad \frac{\partial V_z}{\partial z}\, e_0 = \frac{2}{3}\,t e_0^2\, e_3$

5. Compare the reverse: the product of the spatial derivative of u and the scalar basis vector e_0 and the time derivative of $\vec{V}$ with the 3-vector I:
$\frac{\partial u}{\partial x}\, e_0 = -\,\frac{2}{9}\,x e_0 \quad \frac{\partial V_x}{\partial t}\,\frac{e_1}{3} = \frac{2}{9}\,x e_1$
$\frac{\partial u}{\partial y}\, e_0 = -\,\frac{2}{9}\,y e_0 \quad \frac{\partial V_y}{\partial t}\,\frac{e_2}{3} = \frac{2}{9}\,y e_2$
$\frac{\partial u}{\partial z}\, e_0 = -\,\frac{2}{9}\,z e_0 \quad \frac{\partial V_z}{\partial t}\,\frac{e_3}{3} = \frac{2}{9}\,z e_3$

Note that the basis vectors are *different*, which is the entire reason that the signs are different.

The function f is analytic in q by the Cauchy-Riemann equations.

3.11 Analytic by the Holonomic Equation

The holonomic equation for a quaternion function is:
$\frac{\partial u}{\partial t} + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$

1. Start with the same function f split up the same way:
   $f = q^2$
   $u = t^2 e_0^2 + x^2 \frac{e_1^2}{9} + y^2 \frac{e_2^2}{9} + z^2 \frac{e_3^2}{9}$
   $\vec{V} = (2 t x e_0 \frac{e_1}{3}, 2 t y e_0 \frac{e_2}{3}, 2 t z e_0 \frac{e_3}{3})$
2. Take the relevant derivatives:
   $\frac{\partial u}{\partial t} = 2 t e_0^2$
   $\frac{\partial V_x}{\partial x} = 2 t e_0 \frac{e_1}{3}$
   $\frac{\partial V_y}{\partial y} = 2 t e_0 \frac{e_2}{3}$
   $\frac{\partial V_z}{\partial z} = 2 t e_0 \frac{e_3}{3}$
3. Dot this with the 4-basis vector to form the holonomic equation:
   $(2 t e_0^2, 2 t e_0 \frac{e_1}{3}, 2 t e_0 \frac{e_2}{3}, 2 t e_0 \frac{e_3}{3}).(e_0, e_1, e_2, e_3)$
   $= 2 t e_0^3 + \frac{2}{3} t e_0 e_1^2 + \frac{2}{3} t e_0 e_2^2 + \frac{2}{3} t e_0 e_3^2 = 0$

The function f is analytic in q by the holonomic equation.

3.12 Representing Components with Automorphisms

1. $t = \frac{q + q^*}{2} e_0$ (like real for z).
2. $x = \frac{q + q^{*1}}{-2/3} e_1 = \frac{2/3 x e_1}{-2/3} e_1$
3. $y = \frac{q + q^{*2}}{-2/3} e_2 = \frac{2/3\, y e_2}{-2/3} e_2$
4. $z = \frac{q + q^* + q^{*1} + q^{*2}}{2/3} e_3 = \frac{-2/3\, z e_3}{2/3} e_3$

These are needed to apply the chain rule.

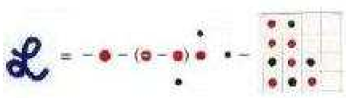
Chapter 4

Unifying Gravity and EM by Analogy to EM: Outline

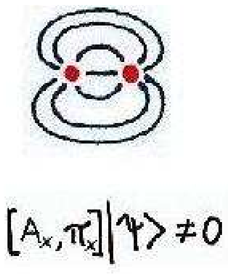
Day 1: Lagrange densities.

Day 2: Fields and quantum mechanics.

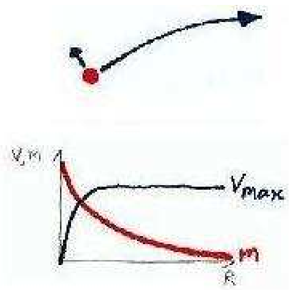
Day 3: Forces, metrics, and new physics.



Day 1



Day 2



Day 3

4.1 Required Skills

- Algebra.
- AP-level calculus.
- Ability to learn *fast*.

Helpful knowledge:

Lagrangians, calculus of variations, complex analysis, dimensional analysis, the Maxwell equations, general relativity, quantum mechanics, perturbation theory, group theory, astrophysics.

$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} - \nabla\times\frac{\partial A}{\partial t} - \nabla\times\nabla\phi = 0 \checkmark$$

4.2 (Title) Information Structure

(Preamble) Definition or explanation.

- (Example 1) Slides.
- (Example 2) Slide summary.
- (Example 3) Hard copy from web at quaternions.com.

1. (Start) Outline or math derivation.
2. (End) Interdependent task completed.

Comment, such as trying to make less than 7 info chunks/slide.

Warning: Visual information may be imprecise!

My apartment looks different.

Slide 57 count to end + random remarks.

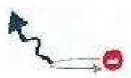
4.3 The Big Picture: A 4D Slinky

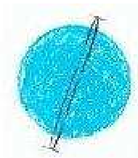
Gravity and light form a slinky in four dimensions (one for time, three for space).

- A slinky wobbles.
- The Earth has wobbled around the Sun 4 billion times.
- Light is created by electrons wobbling.

Want a description of all the interactions in a volume (called a Lagrange density) that can be used to create 4 differential equations (a 4D wave equation). The solutions to those equations must then be linked to the simple harmonic oscillations displayed by gravitational and electronic systems.

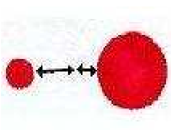
Thought experiment: slow neutrinos could wobble through the Earth act as a SHO, cycling to the other side of the Earth and back every 88 minutes. This is a longitudinal wave, because the acceleration is in the direction of the velocity.

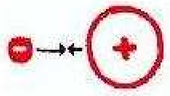


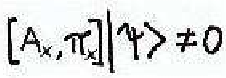


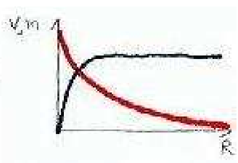
4.4 Must Do Physics

- Gravity.
- EM.
- Quantum mechanics.
- Experimental tests.



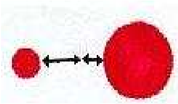


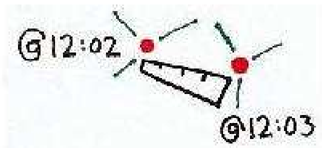




4.4.1 Must Do: Gravity

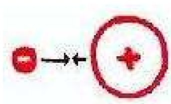
1. $F_g = -Gm\psi \hat{R}$ Like charges attract.
2. $+m$ One charge.
3. $\rho = \nabla^2 \phi$ Newton's gravitational field equation.
4. $m\frac{d^2\vec{R}}{dt^2} = -\frac{GMm}{R^2}\hat{R}$ Newton's law of gravity under classical conditions.
5. $d\tau^2 = (1 - 2\frac{GM}{c^2R} + 2(\frac{GM}{c^2R})^2)dt^2 - (1 + 2\frac{GM}{c^2R})\frac{dR^2}{c^2}$
Consistent with the Schwarzschild metric.

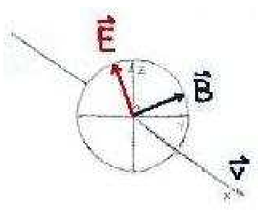




4.4.2 Must Do: Electrodynamics

1. $F_{\text{EM}} = q\vec{E}$ Like charges repel.
2. $\pm q$ Two distinct charges.
3. $\rho = \vec{\nabla} \cdot \vec{E} \quad \vec{J} = -\frac{\partial \vec{E}}{c\partial t} + \vec{\nabla} \times \vec{B}$ Maxwell source equations.
4. $0 = \vec{\nabla} \cdot \vec{B} \quad \vec{0} = \frac{\partial \vec{B}}{c\partial t} + \vec{\nabla} \times \vec{E}$ Maxwell homogeneous equations.
5. $F^{\mu} = q\frac{U_{\mu}}{c}(A^{\mu,\nu} - A^{\nu,\mu})$ Lorentz force.



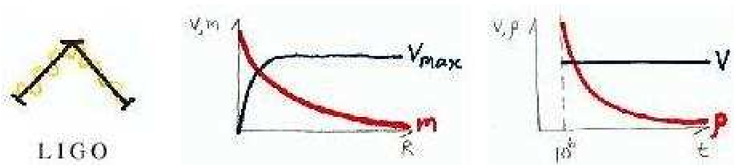


4.4.3 Must Do: Quantum Mechanics

1. Unified field emission modes can be quantized.
2. Works with the standard model.
3. Indicates origin of mass.

4.4.4 Must Do: Experimental Tests

1. LIGO (gravity wave polarization).
2. Rotation profiles of spiral galaxies.
3. Big Bang constant velocity distribution.



4.4.5 Will Not Be Doing

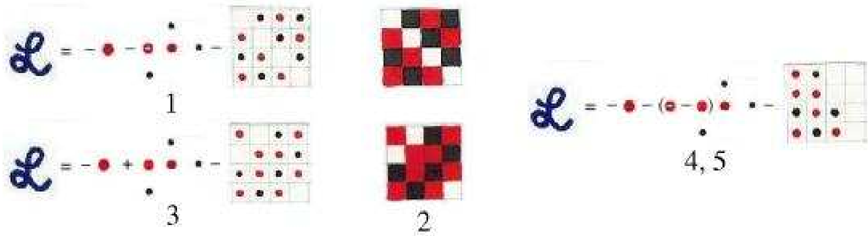
- Review of previous efforts to unify gravity and EM.
- Regenerate Einstein's field equations, $G^{\mu\nu} = 8\pi T^{\mu\nu}$.

Don't bet against Einstein. Einstein viewed general relativity as an intermediate step. The last half of his life was devoted to two tasks: unifying gravity with EM, and understanding why quantum mechanics is the way it is, the logical reason driving it. He was willing to reconstruct physics from the ground up so long as guiding principles were respected. These lectures are devoted to unification. Another lecture series would be required to understand the logic of quantum mechanics, and I do think I know where the answer to that riddle lives.

4.5 Lagrange Densities

Where all mass, energy, and interactions are in a volume.

1. EM Lagrange density.
2. EM to gravity by analogy.
3. Gravity Lagrange density hypothesis.
4. GEM Lagrange density.
5. GEM Lagrange density in detail.



4.5.1 EM Lagrange Densities

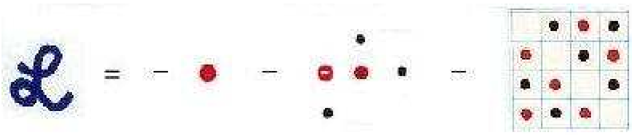
Where all EM energy is in a volume, no gravity.

$$\mathfrak{L}_{\text{EM}} = -\frac{1}{\gamma}\rho_m - \frac{1}{c}J_\mu A^\mu - \frac{1}{4c^2}(\nabla^\mu A^\nu - \nabla^\nu A^\mu)(\nabla_\mu A_\nu - \nabla_\nu A_\mu)$$

- $-\frac{1}{\gamma}\rho_m$ Energy density of mass in motion.
- $-\frac{1}{c}J_\mu A^\mu$ Energy density of electric charge in motion.
- $-\frac{1}{4c^2}(\nabla^\mu A^\nu - \nabla^\nu A^\mu)(\nabla_\mu A_\nu - \nabla_\nu A_\mu)$

  Energy density of antisymmetric change in the potential.

The pattern: rank-0, rank-1 contraction, and rank-2 contraction.

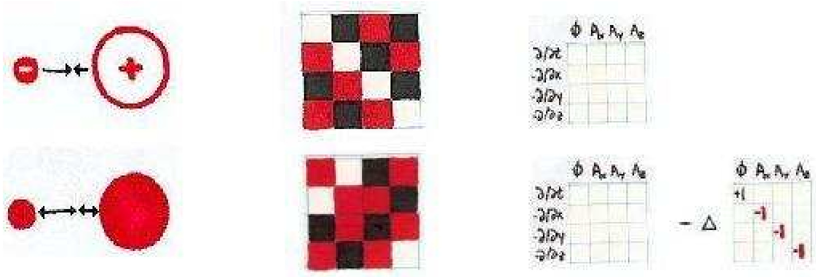


4.5.2 EM to Gravity Analogy

- $-\frac{1}{\gamma}\rho_m$ No change for mass in motion rank-0 term.
- $-q \longrightarrow +\sqrt{G}\,m$ Electric charge to mass charge.
- Change field strength tensor's symmetry.

  $A - A \longrightarrow A + A$ Anti-symmetric to symmetric tensor.

There are two sign changes, both are minus to plus. The first from -q to +m makes the law attractive. The second in the tensor changes the symmetry.



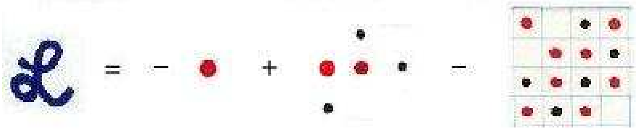
4.5.3 Gravity Lagrange Density Hypothesis

Where all gravitational energy is in a volume, no EM.

$$\mathfrak{L}_G = -\frac{1}{\gamma}\rho_m + \frac{1}{c}J^\mu_m A_\mu - \frac{1}{4c^2}(\nabla^\mu A^\nu + \nabla^\nu A^\mu)(\nabla_\mu A_\nu + \nabla_\nu A_\mu)$$

- $-\frac{1}{\gamma}\rho_m$ Energy density of mass in motion.
- $+\frac{1}{c}J^\mu_m A_\mu$ Energy density of mass charge in motion.
- $-\frac{1}{4c^2}(\nabla^\mu A^\nu + \nabla^\nu A^\mu)(\nabla_\mu A_\nu + \nabla_\nu A_\mu)$
  Energy density of symmetric change in the potential.

Only the rank-1 and rank-2 contraction terms have been changed by the analogy.



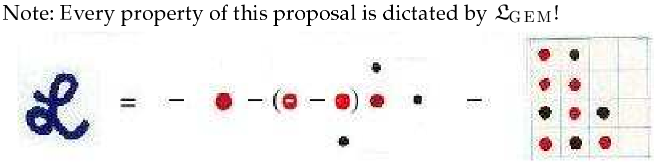
4.5.4 Unified Lagrange Density

$\mathfrak{L}_{\text{GEM}}$ is the union of $\mathfrak{L}_G$ and $\mathfrak{L}_{\text{EM}}$.

- Mass in motion term is a union, not a sum.
- Sum charges in motion terms.
- Sum and simplify field strength tensor terms:
  1. $\nabla^\mu A^\nu \nabla_\nu A_\mu - \nabla^\mu A^\nu \nabla_\nu A_\mu$ Cross terms drop.
  2. $\nabla^\mu A^\nu \nabla_\mu A_\nu = \nabla^\nu A^\mu \nabla_\nu A_\mu$ Contractions are equal.

$$\mathfrak{L}_{\text{GEM}} = -\frac{1}{\gamma}\rho_m - \frac{1}{c}(J^\mu_q - J^\mu_m)A_\mu - \frac{1}{2c^2}\nabla^\mu A^\nu \nabla_\mu A_\nu$$

Note: Every property of this proposal is dictated by $\mathfrak{L}_{\text{GEM}}$!

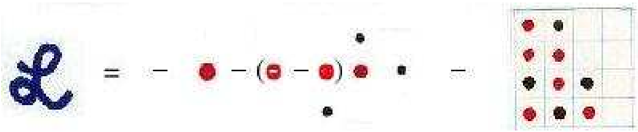


4.5.5 GEM Lagrange Density in Detail

Goal: Get to individual terms, no indices.

Method: Expand, contract, and repeat.

1. Start with the GEM Lagrange density which has $1+4+16$ final terms:
$$\mathfrak{L}_{\text{GEM}} = -\frac{1}{\gamma}\,\rho_m - \frac{1}{c}\,(J_q^\mu - J_m^\mu)A_\mu - \frac{1}{2c^2}\,\nabla^\mu A^\nu\,\nabla_\mu A_\nu$$
2. Expand J^μ, A_μ. Apply the definition of a contravariant derivative to a contravariant vector ($\nabla^\mu A^\nu = \partial^\mu A^\nu - \Gamma_\sigma{}^{\mu\nu}\,A^\sigma$):
$$\mathfrak{L} = -\frac{1}{\gamma}\,\rho_m - (\rho_q - \rho_m)(c, -v^u)(\phi, A^u)$$
$$-\frac{1}{2c^2}(\partial^\mu A^\nu - \Gamma_\sigma{}^{\mu\nu}\,A^\sigma)(\partial_\mu A_\nu - \Gamma^\sigma{}_{\mu\nu}\,A_\sigma)$$
3. Contract U_μ with A^μ. Multiply out final term:
$$\mathfrak{L} = -\frac{1}{\gamma}\,\rho_m - (\rho_q - \rho_m)(c\phi - v^u A^u)$$
$$-\frac{1}{2c^2}(\partial^\mu A^\nu\,\partial_\mu A_\nu - 2\,\Gamma_\sigma{}^{\mu\nu}\,A^\sigma\partial_\mu A_\nu + \Gamma_\sigma{}^{\mu\nu}\,A^\sigma\,\Gamma^\sigma{}_{\mu\nu}\,A_\sigma)$$
4. Expand $\partial_\mu A^\nu$ and $\partial^\mu A_\nu$. Work in local covariant coordinates where $\Gamma = 0$:
$$\mathfrak{L} = -\frac{1}{\gamma}\,\rho_m - (\rho_q - \rho_m)(c\phi - v^u A^u) - \frac{1}{2c^2}(\frac{\partial}{\partial t}, -\nabla^v)(\phi, A^u)(\frac{\partial}{\partial t}, \nabla^v)(\phi, -A^u)$$
5. Contract:
$$\mathfrak{L} = -\frac{1}{\gamma}\,\rho_m - (\rho_q - \rho_m)\,(c\phi - v^u A^u) - \frac{1}{2c^2}((\frac{\partial\phi}{\partial t})^2 - (\nabla\phi)^{v2} - (\frac{\partial A}{\partial t})^{u2} + (\nabla A)^{uv2})$$
6. Write it ALL out:
$$\mathfrak{L} = -\,\rho_m(\sqrt{1 - (\frac{\partial x}{c\partial t})^2 - (\frac{\partial y}{c\partial t})^2 - (\frac{\partial z}{c\partial t})^2} - (\rho_q - \rho_m)(c\phi - \frac{\partial x}{\partial t}A_x - \frac{\partial x}{\partial t}A_y - \frac{\partial z}{\partial t}A_z)$$
$$-\frac{1}{2}((\frac{\partial\phi}{c\partial t})^2 - (\frac{\partial\phi}{\partial x})^2 - (\frac{\partial\phi}{\partial y})^2 - (\frac{\partial\phi}{\partial z})^2 - (\frac{\partial A_x}{c\partial t})^2 + (\frac{\partial A_x}{\partial x})^2 + (\frac{\partial A_x}{\partial y})^2 + (\frac{\partial A_x}{\partial z})^2$$
$$-(\frac{\partial A_y}{c\partial t})^2 + (\frac{\partial A_y}{\partial x})^2 + (\frac{\partial A_y}{\partial y})^2 + (\frac{\partial A_y}{\partial z})^2 - (\frac{\partial A_z}{c\partial t})^2 + (\frac{\partial A_z}{\partial x})^2 + (\frac{\partial A_z}{\partial y})^2 + (\frac{\partial A_z}{\partial z})^2)$$

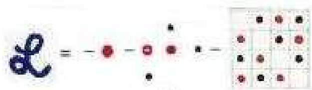


4.5.6 Summary: Lagrange Densities

Math:

$$\mathfrak{L}_{\text{GEM}} = -\frac{1}{\gamma}\,\rho_m - \frac{1}{c}\,(J_q^\mu - J_m^\mu)A_\mu - \nabla^\mu A^\nu\,\nabla_\mu A_\nu$$

Pictures:



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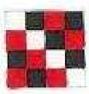

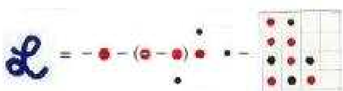

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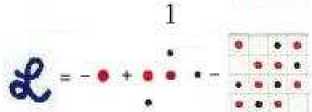

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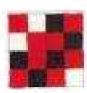
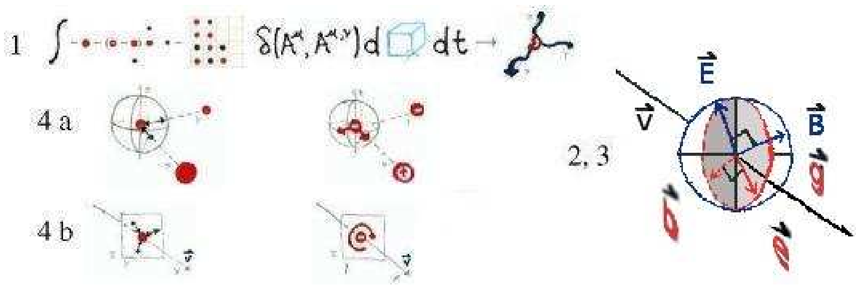


2

4.6 Fields

1. The Players.
2. Euler-Lagrange equation:
    a) Principle of least action.
    b) Derivation.
    c) Apply to GEM Lagrange density.
3. Classical fields.
4. Classical fields in detail.
5. Classical field equations:
    a) Gauss' law and Newton's [relativistic] gravitational field.
    b) Ampere's law and mass current.
    c) Vector identities.



4.6.1 The Players

A table of the players in fields and field equations. Three new fields for gravity will be introduced subsequently.

Rank	*Symbol*	*Name*
0	$\mathfrak{L}$	Lagrange density
1	A^ν	Potential
1	$c\frac{\partial \mathfrak{L}}{\partial A^\nu} = c\nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu})$	Field equations
1	$\frac{\partial \vec{E}}{c\partial t}, \vec{\nabla} \times \vec{E}, \frac{\partial \vec{e}}{c\partial t}, \frac{\partial \vec{B}}{c\partial t}, \vec{\nabla} \times \vec{B}, \vec{\nabla} \boxtimes \vec{b}, \nabla g^\mu$	Field equations as classical fields
2	$\vec{E}, \vec{e}, \vec{B}, \vec{b}, g^\mu$	Classical fields which constitute $\nabla^\mu A^\nu$

4.6.2 Principle of Least Action

The spacetime integral of a Lagrange density:

1. The action:

$S = \int \; \mathfrak{L}\sqrt{-g}\,\partial V \partial t$

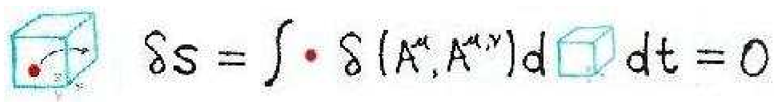
2. Minimize the action:
   $\delta S = \int \; (\frac{\partial \mathfrak{L}}{\partial A^\nu}\delta A^\nu + \frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu}\delta \nabla^\mu A^\nu)\sqrt{-g}\,\partial V \partial t = 0$

Search for minimal function, not value, using calculus of variations.

4.6.3 Derive the Euler-Lagrange Equation

Local covariant coordinates will be used in the following work.

1. Start with a Lagrange density that is a function of A^ν and $\nabla^\mu A^\nu$ (not position or velocity):
   $\mathfrak{L} = f(A^\nu, \nabla^\mu A^\nu)$
2. Form the action:
   $S = \int \; \mathcal{L}(A^\nu, \nabla^\mu A^\nu)\sqrt{-g}\,\partial V \partial t$
3. Take the variation of the action:
   $\delta S = \int \; (\frac{\partial \mathfrak{L}}{\partial A^\nu}\delta A^\nu + \frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu}\delta \nabla^\mu A^\nu)\sqrt{-g}\,\partial V \partial t$
4. Rewrite the 2nd term using the chain rule, subtracting the excess term:
   $\delta S = \int \; (\frac{\partial \mathfrak{L}}{\partial A^\nu}\delta A^\nu + \nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu}\delta A^\nu) - \nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu})\delta A^\nu)\sqrt{-g}\,\partial V \partial t$
5. Integral of 2nd term is zero (a theorem of Gauss):
   $\delta S = \int \; (\frac{\partial \mathfrak{L}}{\partial A^\nu} - \nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu}))\delta A^\nu\sqrt{-g}\,\partial V \partial t$
6. Set integral to zero, which is true for all possible variations if integrand is zero:
   $\frac{\partial \mathfrak{L}}{\partial A^\nu} = \nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu})$



4.6.4 Apply Euler-Lagrange to GEM Lagrange Density

1. Start with Euler-Lagrange, $\frac{\partial \mathfrak{L}}{\partial A^\nu} = \nabla^\mu(\frac{\partial \mathfrak{L}}{\partial \nabla^\mu A^\nu})$, written without indices:

$$c\frac{\partial \mathfrak{L}}{\partial \phi} = c(\frac{\partial}{\partial t}(\frac{\partial \mathfrak{L}}{\partial(\frac{\partial \phi}{\partial t})}) - \frac{\partial}{\partial x}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial \phi}{\partial x})}) - \frac{\partial}{\partial y}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial \phi}{\partial y})}) - \frac{\partial}{\partial z}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial \phi}{\partial z})}))$$

$$c\frac{\partial \mathfrak{L}}{\partial A_x} = c(\frac{\partial}{\partial t}(\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A_x}{\partial t})}) - \frac{\partial}{\partial x}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_x}{\partial x})}) - \frac{\partial}{\partial y}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_x}{\partial y})}) - \frac{\partial}{\partial z}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_x}{\partial z})}))$$

$$c\frac{\partial \mathfrak{L}}{\partial A_y} = c(\frac{\partial}{\partial t}(\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A_y}{\partial t})}) - \frac{\partial}{\partial x}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_y}{\partial x})}) - \frac{\partial}{\partial y}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_y}{\partial y})}) - \frac{\partial}{\partial z}(\frac{\partial \mathfrak{L}}{\partial(-\frac{\partial A_y}{\partial z})}))$$

$$c\,\frac{\partial\mathfrak{L}}{\partial A_z} = c\left(\frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial(\frac{\partial A_z}{\partial t})}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathfrak{L}}{\partial(-\frac{\partial A_z}{\partial x})}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\mathfrak{L}}{\partial(-\frac{\partial A_z}{\partial y})}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\mathfrak{L}}{\partial(-\frac{\partial A_z}{\partial z})}\right)\right)$$

2. Write out GEM Lagrange density without indices:

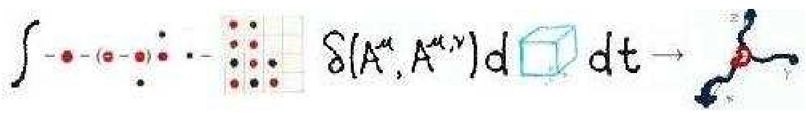
$$\begin{aligned}\mathfrak{L} = &-\rho_m\left(\sqrt{1-(\frac{\partial x}{c\partial t})^2-(\frac{\partial y}{c\partial t})^2-(\frac{\partial z}{c\partial t})^2}\right.\\ &-(\rho_q-\rho_m)(c\phi-\frac{\partial x}{\partial t}A_x-\frac{\partial x}{\partial t}A_y-\frac{\partial z}{\partial t}A_z)\\ &-\frac{1}{2}((\frac{\partial\phi}{c\partial t})^2-(\frac{\partial\phi}{\partial x})^2-(\frac{\partial\phi}{\partial y})^2-(\frac{\partial\phi}{\partial z})^2-(\frac{\partial A_x}{c\partial t})^2+(\frac{\partial A_x}{\partial x})^2+(\frac{\partial A_x}{\partial y})^2+(\frac{\partial A_x}{\partial z})^2\\ &-(\frac{\partial A_y}{c\partial t})^2+(\frac{\partial A_y}{\partial x})^2+(\frac{\partial A_y}{\partial y})^2+(\frac{\partial A_y}{\partial z})^2-(\frac{\partial A_z}{c\partial t})^2+(\frac{\partial A_z}{\partial x})^2+(\frac{\partial A_z}{\partial y})^2+(\frac{\partial A_z}{\partial z})^2)\end{aligned}$$

3. Apply:

$$\begin{aligned}&-(\rho_q-\rho_m) = -\frac{\partial^2\phi}{c\partial t^2}+c\frac{\partial^2\phi}{\partial x^2}+c\frac{\partial^2\phi}{\partial y^2}+c\frac{\partial^2\phi}{\partial z^2}\\ &(\rho_q-\rho_m)\,\frac{\partial x}{c\partial t} = \frac{\partial^2 A_x}{c\partial t^2}-c\frac{\partial^2 A_x}{\partial x^2}-c\frac{\partial^2 A_x}{\partial y^2}-c\frac{\partial^2 A_x}{\partial z^2}\\ &(\rho_q-\rho_m)\,\frac{\partial y}{c\partial t} = \frac{\partial^2 A_y}{c\partial t^2}-c\frac{\partial^2 A_y}{\partial x^2}-c\frac{\partial^2 A_y}{\partial y^2}-c\frac{\partial^2 A_y}{\partial z^2}\\ &(\rho_q-\rho_m)\,\frac{\partial z}{c\partial t} = \frac{\partial^2 A_z}{c\partial t^2}-c\frac{\partial^2 A_z}{\partial x^2}-c\frac{\partial^2 A_z}{\partial y^2}-c\frac{\partial^2 A_z}{\partial z^2}\end{aligned}$$

4. Executive summary:

$$J_q^\nu - J_m^\nu = \Box^2 A^\nu$$



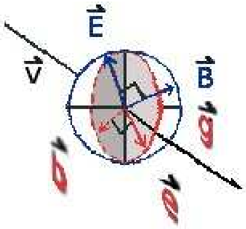
4.6.5 Classical Fields

The classical fields $\vec{E}$ and $\vec{B}$ make up the antisymmetric tensor $(\nabla^\mu A^\nu - \nabla^\mu A^\mu)$. Introduce three new fields, $\vec{e}$ and $\vec{b}$ which have EM counterparts, and a 4-vector field $\boldsymbol{g^\mu}$ for the diagonal components of the symmetric tensor $(\nabla^\mu A^\nu + \nabla^\nu A^\mu)$.

- $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - c\vec{\nabla}\phi$ Electric field.
- $\vec{e} = \frac{\partial\vec{A}}{\partial t} - c\vec{\nabla}\phi - 2\Gamma_\sigma{}^{0i}A^\sigma$ Symmetric analog to electric field.
- $\vec{B} = c\vec{\nabla}\times\vec{A}$ Magnetic field.
- $\vec{b} = -\partial^i A^j - \partial^j A^i - 2\Gamma_\sigma{}^{ij}A^\sigma$
  $= c(0, -\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - 2\Gamma_\sigma{}^{yz}A^\sigma, -\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - 2\Gamma_\sigma{}^{xz}A^\sigma,$
  $-\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - 2\Gamma_\sigma{}^{xy}A^\sigma) \equiv c\vec{\nabla}\boxtimes\vec{A}$
  Symmetric analog to magnetic field.
- $g^\mu = \partial^\mu A^\mu - \Gamma_\sigma{}^{\mu\mu}A^\sigma$
  $= (\frac{\partial\phi}{\partial t} - \Gamma_\sigma{}^{tt}A^\sigma, -c\frac{\partial A_x}{\partial x} - \Gamma_\sigma{}^{xx}A^\sigma, -c\frac{\partial A_y}{\partial y} - \Gamma_\sigma{}^{yy}A^\sigma, -c\frac{\partial A_z}{\partial z} - \Gamma_\sigma{}^{zz}A^\sigma)$
  Diagonal of $\nabla^\mu A^\nu$.

3+3+3+3+4=16 fields total.

All three g's transform differently than axial or polar vectors.



4.6.6 Classical Fields in Detail

1. Start with the asymmetric field strength tensor, $\nabla^{\mu} A^{\nu}$, written as a matrix:

$$\begin{array}{cccc}
\mu = \phi & \mu = A_x & \mu = A_y & \mu = A_z \\
\frac{\partial \phi}{\partial t} - \Gamma_{\sigma}{}^{tt} A^{\sigma} & \frac{\partial A_x}{\partial t} - \Gamma_{\sigma}{}^{tx} A^{\sigma} & \frac{\partial A_y}{\partial t} - \Gamma_{\sigma}{}^{ty} A^{\sigma} & \frac{\partial A_z}{\partial t} - \Gamma_{\sigma}{}^{tz} A^{\sigma} \\
-c\frac{\partial \phi}{\partial x} - \Gamma_{\sigma}{}^{xt} A^{\sigma} & -c\frac{\partial A_x}{\partial x} - \Gamma_{\sigma}{}^{xx} A^{\sigma} & -c\frac{\partial A_y}{\partial x} - \Gamma_{\sigma}{}^{xy} A^{\sigma} & -c\frac{\partial A_z}{\partial x} - \Gamma_{\sigma}{}^{zz} A^{\sigma} \\
-c\frac{\partial \phi}{\partial y} - \Gamma_{\sigma}{}^{yt} A^{\sigma} & -c\frac{\partial A_x}{\partial y} - \Gamma_{\sigma}{}^{yx} A^{\sigma} & -c\frac{\partial A_y}{\partial y} - \Gamma_{\sigma}{}^{\mathrm{yy}} A^{\sigma} & -c\frac{\partial A_z}{\partial y} - \Gamma_{\sigma}{}^{yz} A^{\boldsymbol{\sigma}} \\
-c\frac{\partial \phi}{\partial z} - \Gamma_{\sigma}{}^{zt} A^{\sigma} & -c\frac{\partial A_x}{\partial z} - \Gamma_{\sigma}{}^{zx} A^{\sigma} & -c\frac{\partial A_y}{\partial z} - \Gamma_{\sigma}{}^{zy} A^{\sigma} & -c\frac{\partial A_z}{\partial z} - \Gamma_{\sigma}{}^{zz} A^{\sigma}
\end{array}$$

2. An antisymmetric and symmetric sum equal to $2\nabla^{\mu} A^{\nu}$:

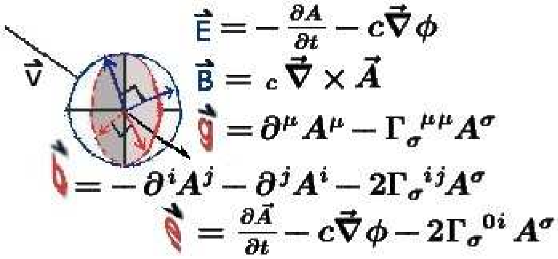
$$\nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu} = \begin{array}{cccc}
0 & \frac{\partial A_x}{\partial t} + c\frac{\partial \phi}{\partial x} & \frac{\partial A_y}{\partial t} + c\frac{\partial \phi}{\partial y} & \frac{\partial A_z}{\partial t} + \mathbf{c}\frac{\partial \phi}{\partial z} \\
-c\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} & 0 & -c\frac{\partial A_y}{\partial x} + c\frac{\partial A_x}{\partial y} & -c\frac{\partial A_z}{\partial x} + c\frac{\partial A_x}{\partial z} \\
-c\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} & -c\frac{\partial A_x}{\partial y} + c\frac{\partial A_y}{\partial x} & 0 & -c\frac{\partial A_z}{\partial y} + c\frac{\partial A_y}{\partial z} \\
-c\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} & -c\frac{\partial A_x}{\partial z} + c\frac{\partial A_z}{\partial x} & -c\frac{\partial A_y}{\partial z} + c\frac{\partial A_z}{\partial y} & 0
\end{array}$$

$$\nabla^{\mu} A^{\nu} + \nabla^{\nu} A^{\mu} =$$

$$\begin{array}{cccc}
2\frac{\partial \phi}{\partial t} - 2\Gamma_{\sigma}{}^{tt} A^{\sigma} & \frac{\partial A_x}{\partial t} - c\frac{\partial \phi}{\partial x} - 2\Gamma_{\sigma}{}^{tx} A^{\sigma} & \frac{\partial A_y}{\partial t} - c\frac{\partial \phi}{\partial y} - 2\Gamma_{\sigma}{}^{ty} A^{\sigma} & -\frac{\partial A_z}{\partial t} - c\frac{\partial \phi}{\partial z} - 2\Gamma_{\sigma}{}^{tz} A^{\sigma} \\
-c\frac{\partial \phi}{\partial x} + \frac{\partial A_x}{\partial t} - 2\Gamma_{\sigma}{}^{xt} A^{\sigma} & 2c\frac{\partial A_x}{\partial x} - 2\Gamma_{\sigma}{}^{xx} A^{\sigma} & -c\frac{\partial A_y}{\partial x} - c\frac{\partial A_x}{\partial y} - 2\Gamma_{\sigma}{}^{xy} A^{\sigma} & -c\frac{\partial A_z}{\partial x} - c\frac{\partial A_x}{\partial z} - 2\Gamma_{\sigma}{}^{xz} A^{\sigma} \\
-c\frac{\partial \phi}{\partial y} + \frac{\partial A_y}{\partial t} - 2\Gamma_{\sigma}{}^{yt} A^{\sigma} & -c\frac{\partial A_x}{\partial y} - c\frac{\partial A_y}{\partial x} - 2\Gamma_{\sigma}{}^{yx} A^{\sigma} & -2c\frac{\partial A_y}{\partial y} - 2\Gamma_{\sigma}{}^{yy} A^{\sigma} & -c\frac{\partial A_z}{\partial y} - c\frac{\partial A_y}{\partial z} - 2\Gamma_{\sigma}{}^{yz} A^{\sigma} \\
-c\frac{\partial \phi}{\partial z} + \frac{\partial A_z}{\partial t} - 2\Gamma_{\sigma}{}^{zt} A^{\sigma} & -c\frac{\partial A_x}{\partial z} - c\frac{\partial A_z}{\partial x} - 2\Gamma_{\sigma}{}^{zx} A^{\sigma} & -c\frac{\partial A_y}{\partial z} - c\frac{\partial A_z}{\partial y} - 2\Gamma_{\sigma}{}^{zy} A^{\sigma} & -2c\frac{\partial A_z}{\partial z} - 2\Gamma_{\sigma}{}^{zz} A^{\sigma}
\end{array}$$

3. $\nabla^{\mu} A^{\nu}$ written in terms of the gravitational, electric, and magnetic fields:

$$\begin{array}{cccc}
g_t & e_x - E_x & e_y - E_y & e_z - E_z \\
e_x + E_x & g_x & b_z - B_z & b_y + B_y \\
e_y + E_y & b_z + B_z & g_y & b_x - B_x \\
e_z + E_z & b_y - B_y & b_x + B_x & g_z
\end{array}$$



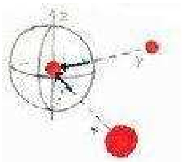
4.6.7 Gauss' Law and Newton's Gravitational Field

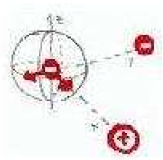
Method: $\frac{1}{2}$ (EM law + gravitational analog) + diagonal terms = field equations.

$$\begin{aligned}\rho_q - \rho_m &= \frac{\partial^2\phi}{c\,\partial t^2} - c\frac{\partial^2\phi}{\partial x^2} - c\frac{\partial^2\phi}{\partial y^2} - c\frac{\partial^2\phi}{\partial z^2}\\
&= \frac{\partial^2\phi}{c\,\partial t^2} + \frac{1}{2}\frac{\partial}{\partial x}((-\frac{\partial A_x}{\partial t} - c\frac{\partial\phi}{\partial x}) + (\frac{\partial A_x}{\partial t} - c\frac{\partial\phi}{\partial x}))\\
&\quad + \frac{1}{2}\frac{\partial}{\partial y}((-\frac{\partial A_y}{\partial t} - c\frac{\partial\phi}{\partial y}) + (\frac{\partial A_y}{\partial t} - c\frac{\partial\phi}{\partial y}))\\
&\quad + \frac{1}{2}\frac{\partial}{\partial z}((-\frac{\partial A_z}{\partial t} - c\frac{\partial\phi}{\partial z}) + (\frac{\partial A_z}{\partial t} - c\frac{\partial\phi}{\partial z}))\\
&= \frac{\partial g_t}{c\partial t} + \frac{1}{2}(\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{e}\,)\end{aligned}$$

- Newton's [relativistic] gravitational field equation results in the physical situation where there is no electric charge density and no divergence of the field $\vec{E}$.
- Gauss' law results in the physical situation with no mass density and no divergence of the field $\vec{e}$.

Implications for forces: Newton's field law implies an attractive force for mass, while Gauss' law indicates like electric charges repulse.





4.6.8 Ampere's Law and Mass Current

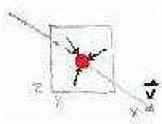
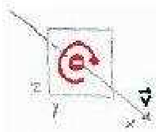
Method: Same as previous.

$$\begin{aligned}\vec{J}_q - \vec{J}_m &= (\frac{\partial^2 A_x}{c\,\partial t^2} - c\frac{\partial^2 A_x}{\partial x^2} - c\frac{\partial^2 A_x}{\partial y^2} - c\frac{\partial^2 A_x}{\partial z^2}, \frac{\partial^2 A_y}{c\,\partial t^2} - c\frac{\partial^2 A_y}{\partial x^2} - c\frac{\partial^2 A_y}{\partial y^2} - c\frac{\partial^2 A_y}{\partial z^2},\\
&\quad \frac{\partial^2 A_z}{c\,\partial t^2} - c\frac{\partial^2 A_z}{\partial x^2} - c\frac{\partial^2 A_z}{\partial y^2} - c\frac{\partial^2 A_z}{\partial z^2})\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}(-\frac{\partial}{\partial t}((-\frac{\partial A_x}{c\,\partial t} - \frac{\partial\,\phi}{\partial x}) - (\frac{\partial A_x}{c\,\partial t} - \frac{\partial\,\phi}{\partial x})) - \frac{\partial}{\partial x}\frac{\partial A_x}{\partial x}\\
&- \frac{\partial}{\partial y}(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}) + \frac{\partial}{\partial y}(-\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}) + \frac{\partial}{\partial z}(-\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}),\\
&- \frac{\partial}{\partial t}((-\frac{\partial A_y}{c\,\partial t} - \frac{\partial\,\phi}{\partial y}) - (\frac{\partial A_y}{c\,\partial t} - \frac{\partial\,\phi}{\partial y})) - \frac{\partial}{\partial y}\frac{\partial A_y}{\partial y}\\
&- \frac{\partial}{\partial z}(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}) + \frac{\partial}{\partial x}(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}) + \frac{\partial}{\partial z}(-\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}) + \frac{\partial}{\partial x}(-\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}),\\
&- \frac{\partial}{\partial t}((-\frac{\partial A_z}{c\,\partial t} - \frac{\partial\,\phi}{\partial z}) - (\frac{\partial A_z}{c\,\partial t} - \frac{\partial\,\phi}{\partial z})) - \frac{\partial}{\partial z}\frac{\partial A_z}{\partial z}\end{aligned}$$

$$-\frac{\partial}{\partial x}(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y})+\frac{\partial}{\partial x}(-\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z})+\frac{\partial}{\partial y}(-\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}))$$
$$=\frac{1}{2}(-\frac{\partial \vec{E}}{c\,\partial t}+\frac{\partial\,\vec{e}}{c\,\partial t}+\vec{\nabla}\times\vec{B}-\nabla\boxtimes\vec{b})+\vec{\nabla}^u g^u$$

- A pure mass current equation results in the physical situation where there is no electric current density no time change of the field $\vec{\boldsymbol{E}}$ and no curl of the field $\vec{\boldsymbol{B}}$.
- Ampere's law results in the physical situation where there is no mass current density, no gradient of the field $\boldsymbol{g^u}$ and not boxed curl of $\vec{\boldsymbol{b}}$.

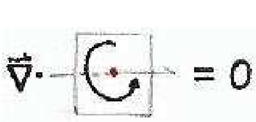



4.6.9 Vector Identities

Vector identities or homogeneous equations are unchanged.

- No magnetic monopoles:
  $\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(c\vec{\nabla}\times\vec{A})=0$
- Faraday's law:
  $\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}-\vec{\nabla}\times\frac{\partial \vec{A}}{\partial t}-\vec{\nabla}\times c\vec{\nabla}\phi=\vec{0}$

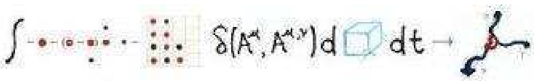
No obvious vector identity analogs for gravitational fields found yet.

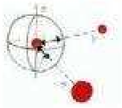


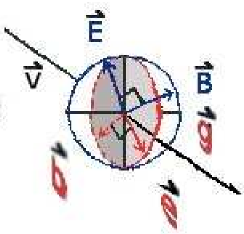

4.6.10 Summary: Field Equations

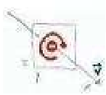
Math:
$J^\mu=\Box^2 A^\mu$

Pictures:

1 

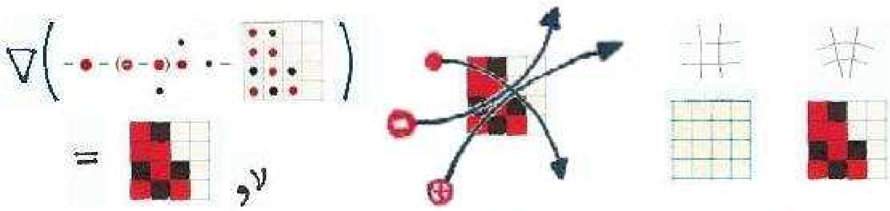
4 a  

2, 3 

4 b  

4.7 Stresses, Forces, and Geodesics

1. Stresses:
   a) Hamiltonian density.
   b) Stress tensor.
   c) Stress tensor of GEM.
2. Forces:
   a) EM Lorentz force.
   b) EM to gravity analogy.
   c) Gravitational force.
   d) GEM force.
3. Geodesics:
   a) Effect of a geodesic.
   b) Cause of curvature in a geodesic.
   c) Killing's differential equation.



1 2 3

4.7.1 The Hamiltonian Density

The Hamiltonian density is a way to characterize the energy in a volume. It can generalized to form the stress tensor which has energy, momentum, and stress all in one tensor.

1. Start from the equation for the Hamiltonian density:
   $\mathfrak{H} = \pi^{\mu} \frac{\partial A_{\mu}}{c\partial t} - \mathfrak{L}$
2. Recall the GEM Lagrange density with no current:
   $\mathfrak{L} = \frac{1}{2}(- (\frac{\partial \phi}{c\partial t})^2 + (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2 + (\frac{\partial A_x}{c\partial t})^2 - (\frac{\partial A_x}{\partial x})^2 - (\frac{\partial A_x}{\partial y})^2 - (\frac{\partial A_x}{\partial z})^2$
   $+ (\frac{\partial A_y}{c\partial t})^2 - (\frac{\partial A_y}{\partial x})^2 - (\frac{\partial A_y}{\partial y})^2 - (\frac{\partial A_y}{\partial z})^2 + (\frac{\partial A_z}{c\partial t})^2 - (\frac{\partial A_z}{\partial x})^2 - (\frac{\partial A_z}{\partial y})^2 - (\frac{\partial A_z}{\partial z})^2)$
3. Calculate the canonical momentum density:
   $\pi^{\mu} = \frac{\partial \mathfrak{L}}{\partial(\frac{\partial A_{\mu}}{c\partial t})} = - \frac{\partial \phi}{c\partial t} - \frac{\partial A_x}{c\partial t} - \frac{\partial A_y}{c\partial t} - \frac{\partial A_x}{c\partial t} = - \frac{\partial A^{\mu}}{c\partial t}$
4. Substitute the momentum π^{μ} into the Hamiltonian density $\mathfrak{H}$:
   $\mathfrak{H} = - \frac{\partial A^{\mu}}{c\partial t} \frac{\partial A_{\mu}}{c\partial t} - \mathfrak{L}$

5. Write out the components:

$$\mathfrak{H} = \tfrac{1}{2}\big(-(\tfrac{\partial\phi}{c\partial t})^2 - (\tfrac{\partial\phi}{\partial x})^2 - (\tfrac{\partial\phi}{\partial y})^2 - (\tfrac{\partial\phi}{\partial z})^2 + (\tfrac{\partial A_x}{c\partial t})^2 + (\tfrac{\partial A_x}{\partial x})^2 + (\tfrac{\partial A_x}{\partial y})^2 + (\tfrac{\partial A_x}{\partial z})^2$$
$$+ (\tfrac{\partial A_y}{c\partial t})^2 + (\tfrac{\partial A_y}{\partial x})^2 + (\tfrac{\partial A_y}{\partial y})^2 + (\tfrac{\partial A_y}{\partial z})^2 + (\tfrac{\partial A_z}{c\partial t})^2 + (\tfrac{\partial A_z}{\partial x})^2 + (\tfrac{\partial A_z}{\partial y})^2 + (\tfrac{\partial A_z}{\partial z})^2\big)$$

6. Rearrange:

$$\mathfrak{H} = -\tfrac{1}{2}(\tfrac{\partial\phi}{c\partial t})^2 + \tfrac{1}{2}(\tfrac{\partial A_x}{\partial x})^2 + \tfrac{1}{2}(\tfrac{\partial A_y}{\partial y})^2 + \tfrac{1}{2}(\tfrac{\partial A_z}{\partial z})^2$$
$$+ \tfrac{1}{2}((\tfrac{\partial A_x}{c\partial t})^2 - (\tfrac{\partial\phi}{\partial x})^2) + \tfrac{1}{2}((\tfrac{\partial A_y}{c\partial t})^2 - (\tfrac{\partial\phi}{\partial y})^2) + \tfrac{1}{2}((\tfrac{\partial A_z}{c\partial t})^2 - (\tfrac{\partial\phi}{\partial z})^2)$$
$$+ \tfrac{1}{4}((\tfrac{\partial A_z}{\partial y})^2 - 2\,\tfrac{\partial A_z}{\partial y}\,\tfrac{\partial A_y}{\partial z} + (\tfrac{\partial A_y}{\partial z})^2) + \cdots$$
$$+ \tfrac{1}{4}((\tfrac{\partial A_z}{\partial y})^2 + 2\,\tfrac{\partial A_z}{\partial y}\,\tfrac{\partial A_y}{\partial z} + (\tfrac{\partial A_y}{\partial z})^2) + \cdots$$

7. Recall the definitions of GEM fields:

$$g^{\mu} = c(\tfrac{\partial\phi}{c\partial t}, -\tfrac{\partial A_x}{\partial x}, -\tfrac{\partial A_y}{\partial y}, -\tfrac{\partial A_z}{\partial z})$$
$$\vec{E} = c(-\tfrac{\partial A_x}{c\partial t} - \tfrac{\partial\phi}{\partial x}, -\tfrac{\partial A_y}{c\partial t} - \tfrac{\partial\phi}{\partial y}, -\tfrac{\partial A_z}{c\partial t} - \tfrac{\partial\phi}{\partial z})$$
$$\vec{e} = c(+\tfrac{\partial A_x}{c\partial t} - \tfrac{\partial\phi}{\partial x}, +\tfrac{\partial A_y}{c\partial t} - \tfrac{\partial\phi}{\partial y}, +\tfrac{\partial A_z}{c\partial t} - \tfrac{\partial\phi}{\partial z})$$
$$\vec{B} = c(+\tfrac{\partial A_z}{\partial y} - \tfrac{\partial A_y}{\partial z}, +\tfrac{\partial A_x}{\partial z} - \tfrac{\partial A_z}{\partial x}, +\tfrac{\partial A_y}{\partial x} - \tfrac{\partial A_x}{\partial y})$$
$$\vec{b} = c(-\tfrac{\partial A_z}{\partial y} - \tfrac{\partial A_y}{\partial z}, -\tfrac{\partial A_x}{\partial z} - \tfrac{\partial A_z}{\partial x}, -\tfrac{\partial A_y}{\partial x} - \tfrac{\partial A_x}{\partial y})$$

8. Rewrite the Hamiltonian density in terms of the GEM fields:

$$\mathfrak{H} = \tfrac{1}{c^2}(-\tfrac{1}{2}g_0^2 + \tfrac{1}{2}\vec{g}^{\,2} - \tfrac{1}{2}\vec{E}\,\vec{e} + \tfrac{1}{4}\vec{B}^2 + \tfrac{1}{4}\vec{b}^{\,2})$$

4.7.2 Stress Tensor

The rank-2 stress tensor is related to a derivative of a Lagrange density.

1. Start with a Lagrange density:

$$\mathfrak{L} = f(A_\sigma, \nabla_\mu A_\sigma)$$

2. Take the derivative:

$$\nabla^\nu \mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial A_\sigma}\nabla^\nu A_\sigma + \frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma}\nabla^\nu\nabla_\mu A_\sigma$$

3. Use the Euler-Lagrange equation on the first term, $\frac{\partial\mathfrak{L}}{\partial A_\sigma} = \nabla_\mu(\frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma})$.
   Change the order of partial derivatives in the second term:

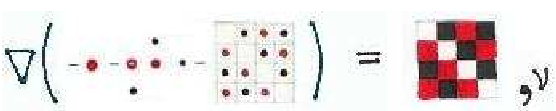
$$\nabla^\nu\mathfrak{L} = \nabla_\mu(\frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma})\nabla^\nu A_\sigma + \frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma}\nabla_\mu\nabla^\nu A_\sigma$$

4. Apply the chain rule to condense into one term:

$$\nabla^\nu\mathfrak{L} = \nabla_\mu((\frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma})\nabla^\nu A_\sigma)$$

5. Define the rank-2 stress tensor as the stuff inside,
   minus the Lagrange density:

$$T^{\mu\nu} \equiv (\frac{\partial\mathfrak{L}}{\partial\nabla_\mu A_\sigma})\nabla^\nu A_\sigma - g^{\mu\nu}\mathcal{L}$$



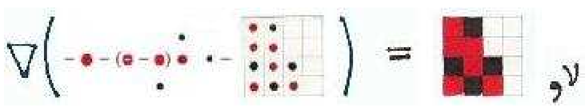
4.7.3 Stress Tensor of GEM

1. Start with the stress tensor definition:
   $T^{\mu\,\nu} \equiv (\frac{\partial \mathfrak{L}}{\partial \nabla_\mu A_\sigma}) \nabla^\nu A_\sigma - g^{\mu\,\nu} \mathcal{L}$
2. GEM Lagrange density in a vacuum:
   $\mathfrak{L}_{\rm GEM} = -\frac{1}{2c^2} \nabla^\lambda A^\sigma \nabla_\lambda A_\sigma$
3. Apply:
   $T^{\mu\,\nu} = -\frac{1}{2c^2} \nabla^{\,\nu} A^\sigma \nabla^\mu A_\sigma + \frac{1}{2c^2} g^{\mu\,\nu} \nabla^\lambda A^\sigma \nabla_\lambda A_\sigma$
4. Write out the energy density term.

$$T^{00} = -\frac{1}{c^2} \frac{\partial A^\sigma}{\partial t} \frac{\partial A_\sigma}{\partial t} + \frac{1}{2c^2} g^{00}((\frac{\partial \phi}{\partial t})^2 - (\nabla \phi)^2 - (\frac{\partial \vec{A}}{\partial t})^2 + (\nabla \vec{A})^2)$$
$$= -\frac{1}{2c^2} (\frac{\partial \phi}{\partial t})^2 + \frac{1}{2c^2} (\frac{\partial \vec{A}}{\partial t})^2 - \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} (\nabla \vec{A})^2$$
$$= -\frac{1}{2} g_0^2 + \frac{1}{2} \vec{g}^{\,2} - \frac{1}{2} e E + \frac{1}{4} B^2 + \frac{1}{4} b^2$$

Notes:

- $T^{00} = E^2 + B^2$ in EM. It is unclear what the difference means.
- $T^{\mu\nu} \longrightarrow F^\mu$ There should be a path between the GEM stress tensor and the relativistic force, but I have not figured it out yet.

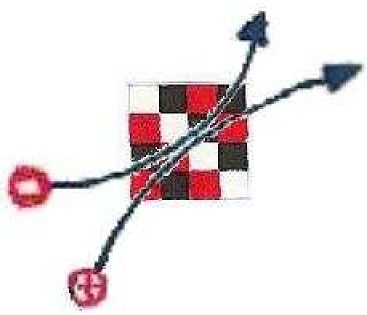


4.7.4 EM Lorentz Force

The Lorentz force is caused by an electric charge moving in an EM field. The effect is to push particles around.

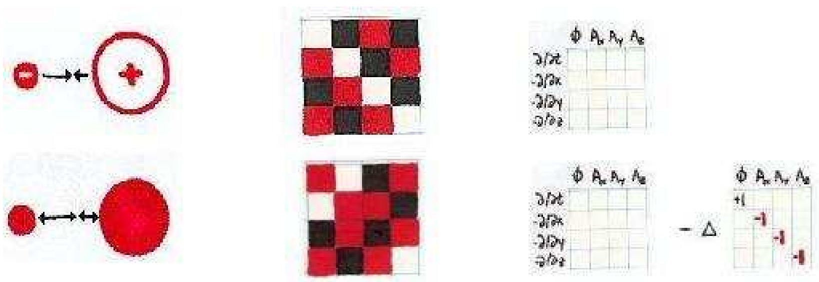
$F^\mu_{\rm EM} = q \frac{U_\nu}{c} (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{\partial m U^\mu}{\partial \tau}$

- The cause is electric charge times the velocity contracted with the antisymmetric field strength tensor.
- The effect is to change momentum with respect to the interval τ.
- If the sign of charge is inverted ($q \longrightarrow -q$), $F^\mu_{\rm EM}$ flips signs, so there are two distinguishable electric charges.
- Like electrical charges are forced away from each other due to the positive sign of the force.



4.7.5 EM to Gravity Analogy

- $-q \longrightarrow +\sqrt{G}\,m$ Electric charge to mass charge.
- Change field strength tensor's symmetry.
  $A - A \longrightarrow A + A$ Anti-symmetric to symmetric tensor.



4.7.6 Gravitational Force

The gravitational force is caused by a mass charge moving in a gravitational field. The effect is to push particles around.

$$F_G^{\mu} = -\sqrt{G}\,m\,\frac{U_{\nu}}{c}(\nabla^{\mu}A^{\nu} + \nabla^{\nu}A^{\mu}) = \frac{\partial m U^{\mu}}{\partial \tau}$$

- The cause is mass charge times the velocity contracted with the symmetric field strength tensor.
- The effect is to change momentum with respect to the interval τ.
- If the sign of mass is inverted ($m \longrightarrow -m$), F_G^{ν} is invariant so there is one distinguishable mass charge.
- Mass charges are forced toward each other due to the negative sign of the force.

4.7.7 GEM Force

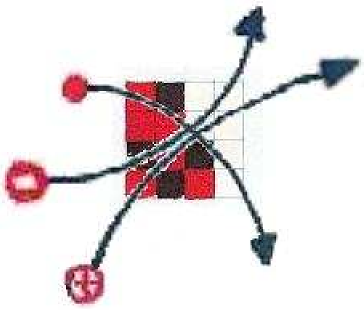
The GEM force is the sum of the gravitational and EM forces.

$$F_{\text{GEM}}^{\mu} = -(\sqrt{G}\,m - q)\,\frac{U_{\nu}}{c}\nabla^{\mu}A^{\nu} - (\sqrt{G}\,m + q)\,\frac{U_{\nu}}{c}\,\nabla^{\nu}A^{\mu} = \frac{\partial m U^{\mu}}{\partial \tau}$$

- $F_{\text{GEM}}^{\nu} = F_G^{\nu}$ if $q = 0$ The GEM force is the gravitational force if the electric charge is zero.
- $F_{\text{GEM}}^{\nu} \longrightarrow F_{\text{EM}}^{\nu}$ as $\frac{\sqrt{G}\,m}{\sqrt{hc}} \longrightarrow 0$

The GEM force approaches the Lorentz force if the mass charge is small compared to the fundamental electric charge ($n\sqrt{hc}$, where n is an integer for the number of quanta of charges). For one electron:

$$\sqrt{\frac{6.67x10^{-11}\,\frac{m^3}{\text{kg}\,s^2}}{6.63x10^{-34}\,\frac{\text{kg}\,m^2}{s}\,3.00x10^8\,\frac{m}{s}}}\;9.11x10^{-31}\,\text{kg} = 1.67x10^{-23}$$

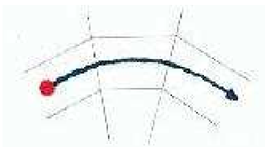


4.7.8 Effect of a Geodesic

A geodesic is the path of zero external force. Investigate the change in momentum (or effect) term of F^{ν}_{GEM}.

1. Start with the change in momentum set equal to zero. Apply the chain rule to expand:
   $$0 = \frac{\partial m U^{\nu}}{\partial \tau} = m\,\frac{\partial U^{\nu}}{\partial \tau} + U^{\nu}\,\frac{\partial m}{\partial \tau}$$
2. Assume $\frac{\partial m}{\partial \tau} = 0$. Use the chain rule to expand $\frac{\partial U^{\nu}}{\partial \tau}$:
   $$0 = m\,\frac{\partial U^{\nu}}{\partial \tau} = m\,\frac{\partial U^{\nu}}{\partial x^{\mu}}\,\frac{\partial x^{\mu}}{\partial \tau} = m\,\nabla_{\mu}U^{\nu}U^{\mu}$$
3. Apply the definition of a covariant derivative of a contravariant vector (normal derivative + change in the metric, $\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + \Gamma_{\varpi\mu}{}^{\nu}A^{\varpi}$):
   $$0 = m\partial_{\mu}U^{\nu}U^{\mu} + m\Gamma_{\varpi\mu}{}^{\nu}U^{\mu}U^{\omega} = m\,\frac{\partial^2 x^{\nu}}{\partial \tau^2} + m\,\Gamma_{\varpi\mu}{}^{\nu}U^{\mu}U^{\omega}$$

If any acceleration is seen without a force ($m\,\frac{\partial^2 x^{\nu}}{\partial \tau^2} \neq 0, F^{\mu}_{\text{GEM}} = 0$), then the effect is entirely due to the curvature of spacetime ($m\Gamma_{\varpi\mu}{}^{\nu}U^{\mu}U^{\omega} \neq 0$).



4.7.9 Cause of Curvature

Every effect must have a cause. Explore the change in potential (or cause) term.

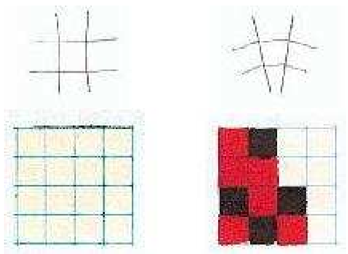
1. Start with force set equal to zero:
   $$0 = -(\sqrt{G}\,m - q)\,\frac{U_{\nu}}{c}\,\nabla^{\mu}A^{\nu} - (\sqrt{G}\,m + q)\,\frac{U_{\nu}}{c}\,\nabla^{\nu}A^{\mu}$$
2. Apply the definition of a covariant derivative of a contravariant vector, normal derivative - change in the metric, $\nabla^{\mu}A^{\nu} = \partial^{\mu}A^{\nu} - \Gamma_{\varpi}{}^{\mu\nu}A^{\varpi}$:

$$\begin{aligned}0 = &-(\sqrt{G}\,m - q)\,\tfrac{U_\nu}{c}\,\partial^\mu A^\nu - (\sqrt{G}\,m + q)\,\tfrac{U_\nu}{c}\,\partial^\nu A^\mu \\ &+ \sqrt{G}\,m\,\tfrac{U_\nu}{c}\,\Gamma_\varpi{}^{\mu\nu}\,A^\varpi - q\,\tfrac{U_\nu}{c}\,\Gamma_\varpi{}^{\mu\nu}\,A^\varpi \\ &+ \sqrt{G}\,m\,\tfrac{U_\nu}{c}\,\Gamma_\varpi{}^{\nu\mu}\,A^\varpi + q\,\tfrac{U_\nu}{c}\,\Gamma_\varpi{}^{\nu\mu}\,A^\varpi \\ = &-(\sqrt{G}\,m - q)\,\tfrac{U_\nu}{c}\,\partial^\mu A^\nu - (\sqrt{G}\,m + q)\,\tfrac{U_\nu}{c}\,\partial^\nu A^\mu + 2\,\sqrt{G}\,m\,\tfrac{U_\nu}{c}\,\Gamma_\varpi{}^{\mu\nu}\,A^\varpi\end{aligned}$$

Curvature is coupled directly to mass, not to q.

Curvature of spacetime without a force ($2\sqrt{G}\,m\,\frac{U_\nu}{c}\,\Gamma_\varpi{}^{\mu\nu}\,A^\varpi \neq 0, F^\mu_{\mathrm{GEM}} = 0$) is caused by change in the potential which are coupled to both the mass charge and electric charge.

General relativity provides a way to calculate curvature by comparing two nearby geodesics using a tidal effect. Because general relativity lacks a means within the geodesic to calculate the cause of curvature, general relativity is incomplete.

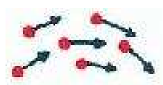


4.7.10 Killing's Differential Equation

If $F^\mu_{\mathrm{GEM}} = 0$, then $\alpha\nabla^\mu A^\nu + \beta\nabla^\nu A^\mu = 0$. This is a generalization of Killing's differential equation where $\alpha = \beta = 1$. The solutions are known as Killing vector fields.

There are two conserved quantities:

- Energy
- Angular momentum

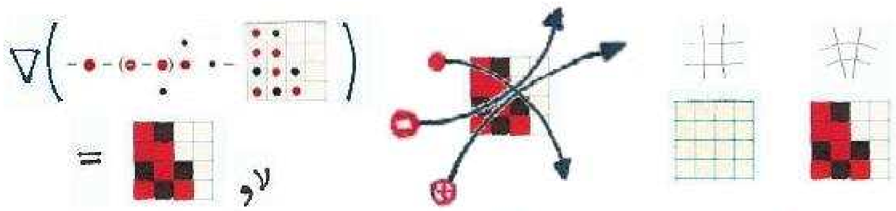


4.7.11 Summary: Stresses, Forces, and Geodesics

Math:

$$F^\mu_{\mathrm{GEM}} = -(\sqrt{G}\,m - q)\,\frac{U_\nu}{c}\,\nabla^\mu A^\nu - (\sqrt{G}\,m + q)\,\frac{U_\nu}{c}\,\nabla^\mu A^\nu = \frac{\partial m U^\mu}{\partial \tau}$$

Pictures:



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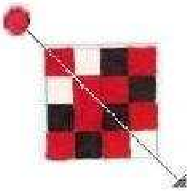
4.8 Relativistic Gravitational Force

1. Weak field approximation.
2. Exact solution.
3. Exact solution applied.
4. Schwarzschild metric.
5. Schwarzschild versus GEM metric.



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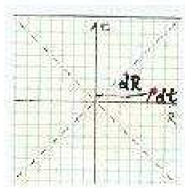

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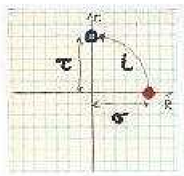

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4.8.1 Weak Field Approximation

1. Start from the gravitational force law:
$$F_G^\mu = -\sqrt{G}\, m \frac{U_\nu}{c}(\nabla^\mu A^\nu + \nabla^\nu A^\mu) = \frac{\partial m U^\mu}{\partial \tau}$$

2. Recall weak gravitational field strength tensor which assumes the field is electrically neutral and weak:
$$\nabla^\mu A^\nu \cong \frac{c^2 k}{\sqrt{G}\,\sigma^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Check units of $A^{\mu,\nu}$ to the derivative of the normalized potential:
$$\sqrt{G}\, m \nabla^\mu A^\nu \rightsquigarrow \frac{\sqrt{L^3}}{t\sqrt{m}}\, m \frac{\sqrt{m}}{t\sqrt{L}} = \frac{mL}{t^2}$$
$$cm\, \frac{\partial \frac{A'^\mu}{|A^\mu|}}{\partial t} \rightsquigarrow \frac{L}{t}\, m\, \frac{1}{t} = \frac{mL}{t^2}$$

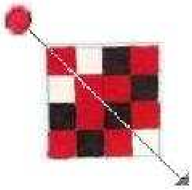
4. Substitute the normalized potential derivative into the force law, noting the units and the sign flip on the contravariant derivative. Expand the velocities, $U_\nu \rightarrow (U_0, -\vec{U})$ and $U^\mu \rightarrow (U_0, \vec{U})$:
$$F_G^\mu = -mc^2\left(\frac{U_0}{c}, -\frac{\vec{U}}{c}\right)\begin{pmatrix} \frac{k}{\sigma^2} & 0 \\ 0 & \frac{k}{\sigma^2} \end{pmatrix} = \left(\frac{\partial m U_0}{\partial \tau}, \frac{\partial m \vec{U}}{\partial \tau}\right)$$

5. Contract the rank-1 velocity tensor with the rank-2 derivative of the potential:
$$F_G^\mu = m\left(-\frac{ck}{\sigma^2} U_0, \frac{ck}{\sigma^2} \vec{U}\right) = \left(\frac{\partial m U_0}{\partial \tau}, \frac{\partial m \vec{U}}{\partial \tau}\right)$$

6. Substitute $c^2\tau^2$ for $-\sigma^2$:

$$F_G^\mu = m(\frac{k}{\tau^2}\frac{U_0}{c}, -\frac{k}{\tau^2}\frac{\vec{U}}{c}) = (\frac{\partial m U_0}{\partial \tau}, \frac{\partial m \vec{U}}{\partial \tau})$$

Warning: The relationship between σ^2 and τ^2 is simple. What gets tricky is the relationship between σ and τ, because there the signs are "free" ($\pm i\sigma \rightarrow \pm c\tau$).



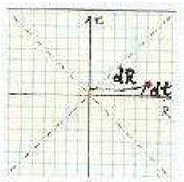
4.8.2 Exact Solution

The gravitational force for the weak field is a first order differential equation that can be solved exactly.

1. Start from the gravitational force for a weak field:
   $$F_G^\mu = m(\frac{k}{\tau^2}\frac{U_0}{c}, -\frac{k}{\tau^2}\frac{\vec{U}}{c}) = (\frac{\partial m U_0}{\partial \tau}, \frac{\partial m \vec{U}}{\partial \tau})$$
2. Apply the chain rule to the cause terms. Assume $U_0 \frac{\partial m}{\partial \tau} = \vec{U}\frac{\partial m}{\partial \tau} = 0$.
   Collect terms on one side:
   $$(m\frac{\partial U_0}{\partial \tau} - m\frac{k}{\tau^2}\frac{U_0}{c}, m\frac{\partial \vec{U}}{\partial \tau} + m\frac{k}{\tau^2}\frac{\vec{U}}{c}) = 0$$
3. Assume the equivalence principle. Drop m:
   $$(\frac{\partial U_0}{\partial \tau} - \frac{k}{\tau^2}\frac{U_0}{c}, \frac{\partial \vec{U}}{\partial \tau} + \frac{k}{\tau^2}\frac{\vec{U}}{c}) = 0$$
4. Solve for velocity:
   $$(U_0, \vec{U}) = (c_0\, e^{-\frac{k}{c\tau}}, \vec{C}_{1-3}\, e^{+\frac{k}{c\tau}})$$
5. Contract the velocity solution:
   $$U^\mu U_\mu = c_0^2 e^{-2\frac{k}{c\tau}} - \vec{C}_{1-3}\, e^{+2\frac{k}{c\tau}}$$
6. For flat spacetime ($k \rightarrow 0$, or $\tau \rightarrow \infty$), there are four constraints on the contracted velocity solution:
   $$U^\mu U_\mu = (c\frac{\partial t}{\partial \tau}, \frac{\partial \vec{R}}{\partial \tau})(c\frac{\partial t}{\partial \tau}, -\frac{\partial \vec{R}}{\partial \tau}) = \frac{c^2(\partial t)^2 - (\partial R)^2}{(\partial t)^2 - (\frac{\partial R}{c})^2} = c^2$$
   True if and only if: $c_0^2 = c\frac{\partial t}{\partial \tau} = U_{0\,\text{flat}}$, $\vec{C}_{1-3} = \frac{\partial \vec{R}}{\partial \tau} = \vec{U}_{\text{flat}}$
7. Substitute $c\frac{\partial t}{\partial \tau}$ for c_0^2, $\frac{\partial \vec{R}}{\partial \tau}$ for $\vec{C}_{1-3}$ into the contracted velocity solution. Multiply through by $(\frac{\partial \tau}{c})^2$:
   $$(\partial \tau)^2 = e^{-2\frac{k}{c\tau}}(\partial t)^2 - e^{+2\frac{k}{c\tau}}(\frac{\partial \vec{R}}{c})^2$$

This is a unique algebraic road to a metric equations. The logic will have to be looked at by mathematicians.

- $k = 0$, or $\tau \rightarrow \infty$ Flat spacetime.
- $e^{-2\frac{k}{c\tau}} \neq 1$ Curved spacetime.



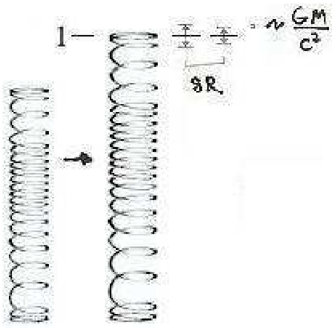
4.8.3 Exact Solution Applied

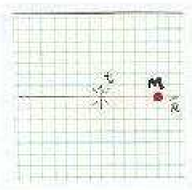
Apply to a weak, spherically symmetric, gravitational system.

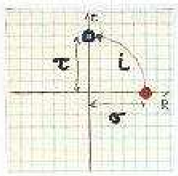
- $k = \frac{GM}{c^2} \rightsquigarrow \frac{L^3}{mt^2} m \frac{t^2}{L^2} = L$ Gravitational source spring constant.
- $\sigma^2 = R^2 - (ct)^2 = R'^2$ Static field approximated by R'.
- $|\sigma| = |c\tau| = R$ σ and $c\tau$ have the same magnitude.
- $(+i\sigma)^2 = (+c\tau)^2$ To make a real metric, choose σ to be imaginary.

Plug into the exact solution:

$$(\partial\tau)^2 = e^{-2\frac{GM}{c^2R}}(\partial t)^2 - e^{+2\frac{GM}{c^2R}}(\frac{\partial\vec{R}}{c})^2$$







4.8.4 Schwarzschild Metric

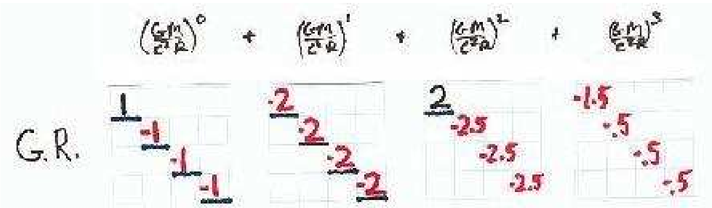
The Schwarzschild metric is a solution of general relativity for a neutral, non-rotating, spherically symmetric source mass (derivation not shown). Write out the Taylor series expansion of the Schwarzschild metric in isotropic coordinates to third order in $\frac{GM}{c^2R}$.

Schwarzschild metric:

$$(\partial\tau)^2 = (\underline{1 - 2\frac{GM}{c^2R} + 2(\frac{GM}{c^2R})^2} - \frac{3}{2}(\frac{GM}{c^2R})^3)(\partial t)^2 - (\underline{1 - 2\frac{GM}{c^2R}} + \frac{3}{2}(\frac{GM}{c^2R})^2 + \frac{1}{2}(\frac{GM}{c^2R})^3)(\frac{\partial\vec{R}}{c})^2$$

The five underlined terms have been confirmed experimentally. Tests include:

- Light bending around the Sun.
- Perihelion shift of Mercury.
- Time delay in radar reflections off of planets.



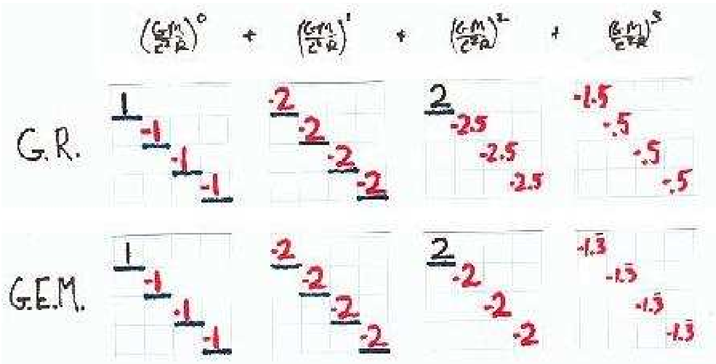
4.8.5 Compare Metrics: Schwarzschild to GEM

Write out the Taylor series expansion of the Schwarzschild and GEM metrics in isotropic coordinates to third order in $\frac{GM}{c^2R}$.

1. Schwarzschild metric:
$$(\partial\tau)^2 = (\underline{1 - 2\frac{GM}{c^2R} + 2(\frac{GM}{c^2R})^2} - \frac{3}{2}(\frac{GM}{c^2R})^3)(\partial t)^2 - (\underline{1 - 2\frac{GM}{c^2R}} + \frac{3}{2}(\frac{GM}{c^2R})^2 + \frac{1}{2}(\frac{GM}{c^2R})^3)(\partial\vec{R})^2$$
2. GEM metric:
$$(\partial\tau)^2 = (\underline{1 - 2\frac{GM}{c^2R} + 2(\frac{GM}{c^2R})^2} - \frac{4}{3}(\frac{GM}{c^2R})^3)(\partial t)^2 - (\underline{1 - 2\frac{GM}{c^2R}} + 2(\frac{GM}{c^2R})^2 + \frac{4}{3}(\frac{GM}{c^2R})^3)(\frac{\partial\vec{R}}{c})^2$$

Compare the two metrics:

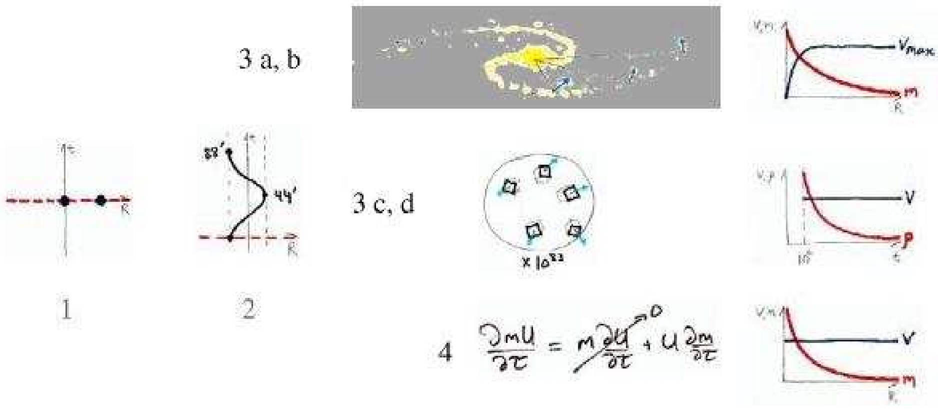
- Identical for tested terms of Taylor series expansion.
- Different for higher order terms, so can be tested (not easy).
- GEM is more symmetric.



4.9 Classical Gravitational Force

1. Breaking spacetime symmetry.
2. Newton's gravitational law derivation.
3. Need for new classical solutions:
    a) Problem statement for rotation profiles of spiral galaxies.
    b) Solution requirements for rotation profiles.
    c) Problem statement for the big bang.
    d) Solution requirements for the big bang.

4. Constant velocity solutions.

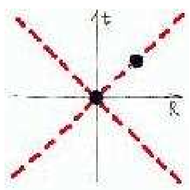


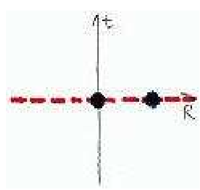
4.9.1 Breaking Spacetime Symmetry

Spacetime symmetry must be broken to go from the relativistic weak gravitational force to a classical force for both cause and effect.

Contrast the relativistic geometry of Minkowski spacetime with the geometry of Newtonian absolute space and time.

Minkowski Spacetime	Geometry	Newtonian Space and Time
True, Elegant	Utility	Accurate, Practical
$(\partial\tau)^2 = (dt)^2 - \frac{(dR)^2}{c^2}$	Interval	$\text{distance}^2 = dR^2 \neq f(t)$
$c = 1$	Speed of Light	$c = \infty$
$(U_0, \vec{U}) = (c\frac{\partial t}{\partial\tau}, \frac{\partial\vec{R}}{\partial\tau})$	Velocity	$(\mathbb{U}_0, \vec{\mathbb{U}}) \equiv (\frac{\partial t}{\partial\|R\|}, c\frac{\partial\vec{R}}{\partial\|R\|}) = (0, c\hat{R})$
$(\frac{\partial U_0}{\partial\tau}, \frac{\partial\vec{U}}{\partial\tau}) = (c\frac{\partial^2 t}{\partial\tau^2}, \frac{\partial^2\vec{R}}{\partial\tau^2})$	Acceleration	$(\frac{\partial\mathbb{U}_0}{\partial\tau}, \frac{\partial\vec{\mathbb{U}}}{\partial\tau}) = (0, c^2\frac{\partial^2\vec{R}}{\partial\|R\|^2})$

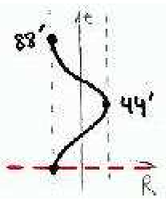




4.9.2 Newton's Gravitational Law Derivation

1. Start from the relativistic gravitational force for a weak field:
   $F_G^\mu = m\,(\frac{k}{\tau^2}\,\frac{U_0}{c}, -\,\frac{k}{\tau^2}\,\frac{\vec{U}}{c}) = (\frac{\partial\, m U_0}{\partial\tau}, \frac{\partial\, m\vec{U}}{\partial\tau})$
2. Apply the chain rule to the cause terms.
   Assume $U_0\,\frac{\partial m}{\partial\tau} = \vec{U}\,\frac{\partial m}{\partial\tau} = 0$:
   $F_G^\mu = m\,(\frac{k}{\tau^2}\,\frac{U_0}{c}, -\,\frac{k}{\tau^2}\,\frac{\vec{U}}{c}) = (m\,\frac{\partial U_0}{\partial\tau}, m\,\frac{\partial\vec{U}}{\partial\tau})$
3. Break spacetime symmetry:
   - $(U_0, \vec{U}) \longrightarrow (\mathbb{U}_0, \vec{\mathbb{U}}) = (0, c\,\hat{R})$
   - $(\frac{\partial U_0}{\partial\tau}, \frac{\partial\vec{U}}{\partial\tau}) \longrightarrow (0, c^2\,\frac{\partial^2\vec{R}}{\partial|R|^2})$

   $F_G^\mu = m\,(0, -\,\frac{k}{\tau^2}\,\hat{R}) = (0, m\,c^2\,\frac{\partial^2\vec{R}}{\partial|R|^2})$
4. Assume the gravitational spring constant ($k = \frac{GM}{c^2}$):
   $F_G^\mu = (0, -\,\frac{GMm}{c^2\tau^2}\,\hat{R}) = (0, m\,c^2\,\frac{\partial^2\vec{R}}{\partial|R|^2})$
5. Substitute: σ^2 for $-\,c^2\,\tau^2$ in the cause term.
   Substitute: $-\,c^2\,(\frac{\partial}{\partial\tau})^2$ for $(\frac{\partial}{\partial|R|})^2 = (\frac{\partial}{\partial\sigma})^2$ in the effect term.
   $F_G^\mu = (0, \frac{GMm}{\sigma^2}\,\hat{R}) = (0, -\,m\,\frac{\partial^2\vec{R}}{\partial\tau^2})$
6. Assume the static field approximation: $\sigma^2 = R^2 - t^2 \cong R'^2$.
   Assume the low speed approximation: $\frac{\partial^2}{\partial\tau^2} \cong \frac{\partial^2}{\partial t^2}$:
   $F_G^\mu = (0, \frac{GMm}{R^2}\,\hat{R}) = (0, -\,m\,\frac{\partial^2\vec{R}}{\partial t^2})$ QED

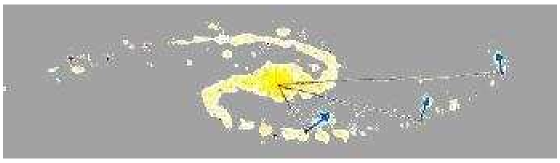


4.9.3 Problem Statement for the Rotation Profile of Galaxies

The momentum of stars in thin spiral galaxies has two problems:

- The flat velocity profile problem.
  After attaining a maximal speed consistent with Newton's law of gravity near the core, the velocity profile stays flat with increasing distance. Newton's law predicts a "Keplerian" decline for the velocity of the outer stars.
- The stability problem.

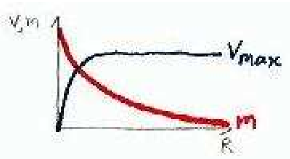
Thin spiral galaxies are mathematically unstable to small disturbances along the axis which should lead to collapse.



4.9.4 Solution Requirements for Rotation Profiles

Requirements for a solution:

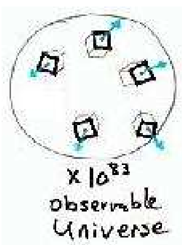
1. Stable mathematically to axial perturbations.
2. Same velocity for all outer stars.
3. Describes the change in mass distribution in spacetime, which falls off exponentially with distance ($2 \times 3 = \triangle$momentum).
4. Fits every observational constraint.



4.9.5 Problem Statement for the Big Bang

Big bang cosmology has two big problems:

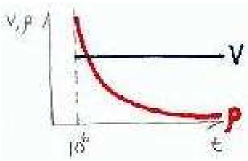
- The horizon problem.
  All $\sim 10^{83}$ separate, independent spacetime volumes of the early Universe must travel at the same velocity to create the uniform black body radiation spectrum seen in the cosmic background radiation.
- The flatness problem.
  The initial conditions must be tuned to one part in $\sim 10^{55}$ so the mathematically unstable solution lasts 10^{10} years.



4.9.6 Solution Requirements for the Big Bang

Requirements for a solution:

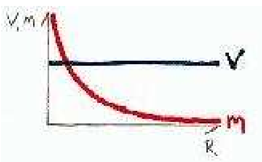
1. Stable mathematically for initial conditions.
2. Same velocity for all independent regions of spacetime.
3. Describes the change in mass distribution in spacetime, from high density early to lower later ($2 \times 3 = \triangle$momentum).
4. Fits every observational constraint.



Disclosure: I do not know the actual shape of mass density decrease.

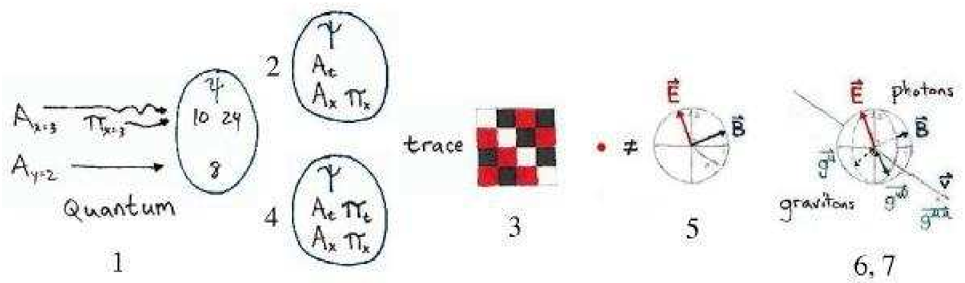
4.9.7 Stable Constant Velocity Solutions

1. Start from the gravitational force for a weak field:
   $F_G^\mu = m(\frac{k}{\tau^2}\frac{U_0}{c}, -\frac{k}{\tau^2}\frac{\vec{U}}{c}) = (\frac{\partial m U_0}{\partial \tau}, \frac{\partial m \vec{U}}{\partial \tau})$
2. Apply the chain rule to the cause terms.
   Assume $m\frac{\partial U_0}{\partial \tau} = m\frac{\partial \vec{U}}{\partial \tau} = 0$ (meaning assume velocity is constant):
   $F_G^\mu = m(\frac{k}{\tau^2}\frac{U_0}{c}, -\frac{k}{\tau^2}\frac{\vec{U}}{c}) = (U_0\frac{\partial m}{\partial \tau}, \vec{U}\frac{\partial m}{\partial \tau})$
3. Break spacetime symmetry: $(U_0, \vec{U}) \longrightarrow (\mathbb{U}_0, \vec{\mathbb{U}}) = (0, c\hat{R})$.
   $F_G^\mu = m(0, -\frac{k}{\tau^2}\hat{R}) = (0, \frac{\partial m}{\partial \tau}c\hat{R})$
4. Assume the gravitational spring constant ($k = \frac{GM}{c^2}$):
   $F_G^\mu = (0, -\frac{GMm}{c^2\tau^2}\hat{R}) = (0, \frac{\partial m}{\partial \tau}c\hat{R})$
5. Collect terms on one side:
   $(c\frac{\partial m}{\partial \tau} + \frac{GMm}{c^2\tau^2})(0, \hat{R}) = 0$
6. Solve for m:
   $m = m_o e^{\frac{GM}{c^3\tau}}$
7. Substitute: R for $c\tau$ which depends on exactly the same assumptions used in the metric derivation (static field, $|\sigma| = |\tau|$=R, and sigma is imaginary):
   $m = m_0 e^{\frac{GM}{c^2 R}}$



4.10 Quantization

1. Classical physics versus quantum mechanics.
2. Momentum from classic EM Lagrange density.
3. Quantizing EM fields by fixing the gauge.
4. Interpreting quantizing EM by fixing the Lorenz gauge.
5. Skeptical analysis of fixing the Lorenz gauge.
6. Momentum from GEM Lagrange density.
7. Interpreting GEM quantization.



4.10.1 Classical Physics versus Quantum Mechanics

Classical physics:

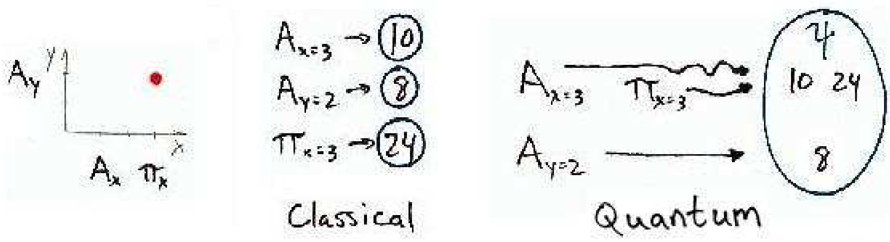
- Observables are numbers
  $A_x = 10 \quad A_y = 8 \quad \pi_x = 24$
- All observables are independent:
  $A_x \, \pi_x - \pi_x \, A_x = 0$

Quantum mechanics:

- Observables are operators that act on
  the wave function ψ to generate a number.
  $A_x|\psi\rangle = 10 \quad A_y|\psi\rangle = 8 \quad \pi_x|\psi\rangle = 24$
- Most observables are independent.
  $[A_x, A_y]$ is called the *commutator*.
  $A_x A_y|\psi\rangle - A_y A_x|\psi\rangle = [A_x, A_y]|\psi\rangle = 0$
- Conjugate observables are *not* independent.

$$[A_x, \pi_x]|\psi\rangle \neq 0$$

Conjugate observables, like the potential and momentum, must have a non-zero commutator to quantize a field.



4.10.2 Momentum from Classic EM Lagrangian

1. Start with the EM Lagrange density written without indices.

$$\mathfrak{L}_{\text{EM}} = -\frac{1}{\gamma}\rho_m - \frac{1}{c}J_\mu A^\mu - \frac{1}{4c^2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\begin{aligned}
= &-\rho_m\Big(\sqrt{1 - (\tfrac{\partial x}{c\partial t})^2 - (\tfrac{\partial y}{c\partial t})^2 - (\tfrac{\partial z}{c\partial t})^2} \\
&-(\rho_q - \rho_m)(c\phi - \tfrac{\partial x}{\partial t}A_x - \tfrac{\partial x}{\partial t}A_y - \tfrac{\partial z}{\partial t}A_z) \\
&-\tfrac{1}{2}\Big(\Big(\quad -(\tfrac{\partial \phi}{\partial x})^2 - (\tfrac{\partial \phi}{\partial y})^2 - (\tfrac{\partial \phi}{\partial z})^2 \\
&-(\tfrac{\partial A_x}{c\partial t})^2 \qquad + (\tfrac{\partial A_x}{\partial y})^2 + (\tfrac{\partial A_x}{\partial z})^2 \\
&-(\tfrac{\partial A_y}{c\partial t})^2 + (\tfrac{\partial A_y}{\partial x})^2 \qquad + (\tfrac{\partial A_y}{\partial z})^2 \\
&-(\tfrac{\partial A_z}{c\partial t})^2 + (\tfrac{\partial A_z}{\partial x})^2 + (\tfrac{\partial A_z}{\partial y})^2 \\
&-2\frac{\partial A_x}{c\partial t}\frac{\partial \phi}{\partial x} - 2\frac{\partial A_y}{c\partial t}\frac{\partial \phi}{\partial y} - 2\frac{\partial A_z}{c\partial t}\frac{\partial \phi}{\partial z} \\
&-2\frac{\partial A_y}{\partial z}\frac{\partial A_z}{\partial y} - 2\frac{\partial A_z}{\partial x}\frac{\partial A_x}{\partial z} - 2\frac{\partial A_x}{\partial y}\frac{\partial A_y}{\partial x}\Big)
\end{aligned}$$

2. Calculate momentum:

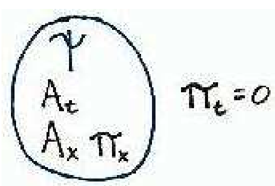
$$\pi^\mu = h\sqrt{G}\,\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A^\mu}{c\partial t})} = h\sqrt{G}\,(0, \frac{\partial A_x}{c\partial t} + \frac{\partial \phi}{\partial x}, \frac{\partial A_y}{c\partial t} + \frac{\partial \phi}{\partial y}, \frac{\partial A_z}{c\partial t} + \frac{\partial \phi}{\partial z})$$

   Energy-momentum vector.

3. Momentum cannot be made into an operator:

$$[A_t, \pi_t]|\psi\rangle = [A_t, 0]|\psi\rangle = 0$$

   Energy commutes with its conjugate operator.

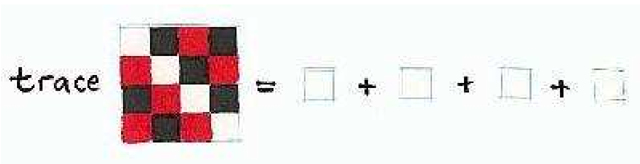


4.10.3 Quantizing EM Fields by Fixing the Gauge

An EM gauge is a relationship between ϕ and $\vec{A}$ that does not change the Maxwell equations. Examples:

- Coulomb gauge.
  $\mathrm{trace}(A^{\mu,\nu}) = \vec{\nabla}\cdot\vec{A} = 0$
- Lorenz gauge.
  $\mathrm{trace}(A^{\mu,\nu}) = \frac{\partial\phi}{c\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

For EM with no gravity, one is free to assign arbitrary values to the diagonal of the antisymmetric field strength tensor.



4.10.4 Quantizing EM by Fixing the Lorenz Gauge

Fix the Lorenz gauge in the EM Lagrange density.

1. Start with the Gupta-Bleuler Lagrange density written without indices:

$$
\begin{aligned}
\mathfrak{L}_{G-B} =& -\frac{1}{\gamma}\rho_m - J_\mu A^\mu - \frac{1}{2c^2}(\partial_\mu A^\mu)^2 \\
& - \frac{1}{4c^2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial A_\mu) \\
=& -\rho_m(\sqrt{1-(\frac{\partial x}{c\partial t})^2-(\frac{\partial y}{c\partial t})^2-(\frac{\partial z}{c\partial t})^2} \\
& -(\rho_q-\rho_m)(c\phi - \frac{\partial x}{\partial t}A_x - \frac{\partial x}{\partial t}A_y - \frac{\partial z}{\partial t}A_z) \\
& -\frac{1}{2}((\frac{\partial\phi}{c\partial t})^2 - (\frac{\partial\phi}{\partial x})^2 - (\frac{\partial\phi}{\partial y})^2 - (\frac{\partial\phi}{\partial z})^2 \\
& -(\frac{\partial A_x}{c\partial t})^2 + (\frac{\partial A_x}{\partial x})^2 + (\frac{\partial A_x}{\partial y})^2 + (\frac{\partial A_x}{\partial z})^2 \\
& -(\frac{\partial A_y}{c\partial t})^2 + (\frac{\partial A_y}{\partial x})^2 + (\frac{\partial A_y}{\partial y})^2 + (\frac{\partial A_y}{\partial z})^2 \\
& -(\frac{\partial A_z}{c\partial t})^2 + (\frac{\partial A_z}{\partial x})^2 + (\frac{\partial A_z}{\partial y})^2 + (\frac{\partial A_z}{\partial z})^2 \\
& -2\frac{\partial A_x}{c\partial t}\frac{\partial\phi}{\partial x} - 2\frac{\partial A_y}{c\partial t}\frac{\partial\phi}{\partial y} - 2\frac{\partial A_z}{c\partial t}\frac{\partial\phi}{\partial z} \\
& -2\frac{\partial A_y}{\partial z}\frac{\partial A_z}{\partial y} - 2\frac{\partial A_z}{\partial x}\frac{\partial A_x}{\partial z} - 2\frac{\partial A_x}{\partial y}\frac{\partial A_y}{\partial x} \\
& +2\frac{\partial\phi}{c\partial t}\frac{\partial A_x}{\partial x} + 2\frac{\partial\phi}{c\partial t}\frac{\partial A_y}{\partial y} + 2\frac{\partial\phi}{c\partial t}\frac{\partial A_z}{\partial z} \\
& +2\frac{\partial A_x}{\partial x}\frac{\partial A_y}{\partial y} + 2\frac{\partial A_x}{\partial x}\frac{\partial A_z}{\partial z} + 2\frac{\partial A_y}{\partial y}\frac{\partial A_z}{\partial z})
\end{aligned}
$$

2. Calculate momentum:

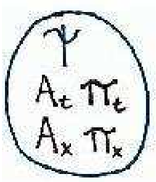
$\pi^{\mu} = h\sqrt{G}\,(-\frac{\partial\phi}{c\partial t} - \vec{\nabla}\cdot\vec{A}\,, \frac{\partial A_x}{c\partial t} + \frac{\partial\phi}{\partial x}, \frac{\partial A_y}{c\partial t} + \frac{\partial\phi}{\partial y}, \frac{\partial A_z}{c\partial t} + \frac{\partial\phi}{\partial z})$
Energy-momentum vector.

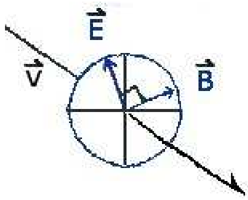
3. Momentum can be made into an operator:

Using the Euler-Lagrange equation [not shown], the equations of motion are identical to those of $\mathfrak{L}_{\mathrm{GEM}}$!

$J^{\mu} = \Box^2 A^{\mu}$

Reference: "Theory of longitudinal photons in quantum electrodynamics", Suraj N. Gupta, Proc. Phys. Soc. 63:681-691, 1950.

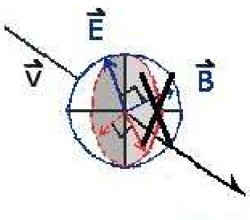




4.10.5 Interpreting the Gupta/Bleuler Quantization Method

Results of quantization method:

- Four modes of transmission:
  1. Two transverse waves.
  2. One longitudinal wave.
  3. One scalar wave.
- Transverse waves are photons for EM.
- Scalar mode of transmission called a "scalar photon".
- "Supplementary condition" imposed to eliminate
  scalar and longitudinal photons as real particles,
  so they are always virtual.



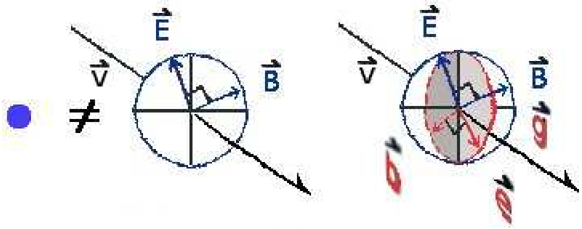
4.10.6 Skeptical Analysis of Fixing the Lorenz Gauge

1. A scalar photon is an oxymoron.

Photons must transform like vectors,
even if photons happen to be virtual.

2. Eliminating an oxymoron
   cannot justify the supplementary condition.

3. A better interpretation for the
   4D-wave equation of motion may exist.



4.10.7 Momentum from GEM Lagrange Density

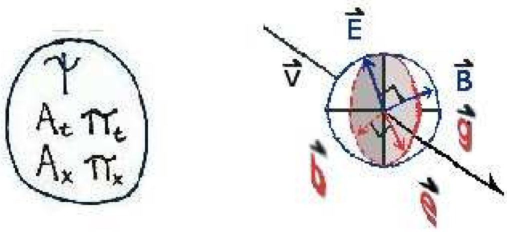
1. Start with the GEM Lagrange density written without indices:

$$
\begin{aligned}
\mathfrak{L}_{\mathrm{GEM}} = &-\frac{1}{\gamma}\,\rho_m - \frac{1}{c}\,(J_q^\mu - J_m^\mu)A_\mu - \frac{1}{2c^2}\,\nabla_\mu A^\nu \nabla^\mu A_\nu \\
= &-\rho_m\Big(\sqrt{1-(\frac{\partial x}{c\partial t})^2-(\frac{\partial y}{c\partial t})^2-(\frac{\partial z}{c\partial t})^2} \\
&-(\rho_q-\rho_m)(c\phi - \frac{\partial x}{\partial t}A_x - \frac{\partial x}{\partial t}A_y - \frac{\partial z}{\partial t}A_z) \\
&-\frac{1}{2}\Big((\frac{\partial \phi}{c\partial t})^2-(\frac{\partial \phi}{\partial x})^2-(\frac{\partial \phi}{\partial y})^2-(\frac{\partial \phi}{\partial z})^2 \\
&-(\frac{\partial A_x}{c\partial t})^2+(\frac{\partial A_x}{\partial x})^2+(\frac{\partial A_x}{\partial y})^2+(\frac{\partial A_x}{\partial z})^2 \\
&-(\frac{\partial A_y}{c\partial t})^2+(\frac{\partial A_y}{\partial x})^2+(\frac{\partial A_y}{\partial y})^2+(\frac{\partial A_y}{\partial z})^2 \\
&-(\frac{\partial A_z}{c\partial t})^2+(\frac{\partial A_z}{\partial x})^2+(\frac{\partial A_z}{\partial y})^2+(\frac{\partial A_z}{\partial z})^2\Big)
\end{aligned}
$$

2. Calculate momentum:

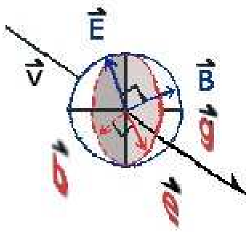
$$
\pi^\mu = h\sqrt{G}\,\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A^\mu}{c\partial t})} = h\sqrt{G}\,(-\frac{\partial \phi}{c\partial t}, \frac{\partial A_x}{c\partial t}, \frac{\partial A_y}{c\partial t}, \frac{\partial A_z}{c\partial t})
$$

3. Momentum can be made into an operator:



4.10.8 GEM Quantization

- Four modes of transmission:
  1. Two transverse waves.
  2. One longitudinal wave.
  3. One scalar wave.
- Transverse waves are photons for EM.
- Longitudinal and scalar modes are gravitons of gravity traveling at the speed of light, generated by a symmetric rank-2 field strength tensor.
- General relativity predicts transverse waves, not scalar or longitudinal ones. The LIGO experiment to detect gravitational waves will be looking for transverse gravitational waves. GEM predicts the polarization will not be transverse.
- Gravitational modes are coupled to $\sqrt{G}$ and not h bar. This might get around negative energy problem because gravity quanta are not emitted.

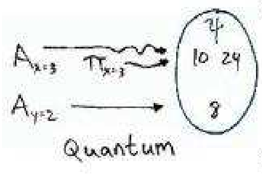


4.10.9 Summary: Quantization

Math:

$$\pi^{\mu} = h\sqrt{G}\,\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A^{\mu}}{c\partial t})} = h\sqrt{G}\,(-\frac{\partial \phi}{c\partial t}, \frac{\partial A_x}{c\partial t}, \frac{\partial A_y}{c\partial t}, \frac{\partial A_z}{c\partial t})$$

Pictures:



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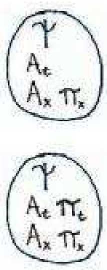

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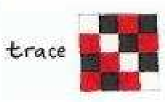

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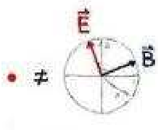

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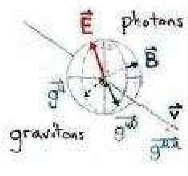

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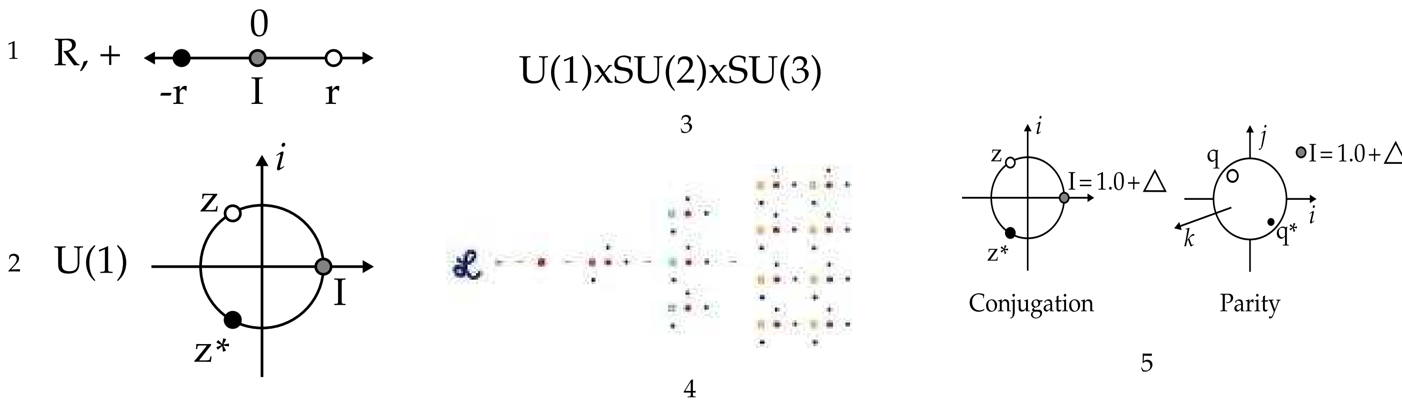
4.11 The Standard Model

1. Group theory.
2. Group theory by example.
3. The standard model.
4. The standard model Lagrange density.
5. Defining the multiplication operator.



4.11.1 Group Theory

Way to organize symmetry systematically.
Definition: A set S with a binary operation ($\times$ or $+$)
such that $s_1 \times s_2 \in S$ for all possible pairs of elements in S. A group has:

- An identity.
- An inverse for every element.
- Associative law holds.

Examples:

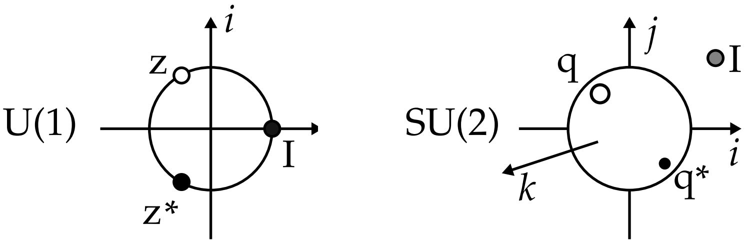
- Real numbers and +.
- Real numbers without 0 and $\times$.

4.11.2 Group Theory by Example

- U(1), $z \times z^* = 1$, or unitary complex numbers.
  - $I = (1, 0)$ Identity is one.

- $z^{-1} = z^*$ Inverse is the conjugate.
- $z_1 \times z_2 = z_2 \times z_1$ Abelian.
- $U(\alpha) = e^{\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}}$ One number for the Lie algebra.

- SU(2), $q \times q^* = 1$, or unit quaternions (4D analog to complex numbers).
  - $I = (1, 0, 0, 0)$ Identity is one.
  - $q^{-1} = q^*$ Inverse is the conjugate.
  - $q_1 \times q_2 \neq q_2 \times q_1$ Non-Abelian.
  - $U(\alpha, \beta, \gamma) = e^{\begin{pmatrix} 0 & -\alpha & -\beta & -\gamma \\ \alpha & 0 & -\gamma & \beta \\ \beta & \gamma & 0 & -\alpha \\ \gamma & -\beta & \alpha & 0 \end{pmatrix}}$ Three numbers needed.



4.11.3 The Standard Model

Predicts patterns of all subatomic particles
and three of four forces in Nature:

- U(1) Light, EM.
- SU(2) Weak force, radioactivity.
- SU(3) Strong force, the nucleus.

Says nothing about gravity.

U(1)xSU(2)xSU(3)

4.11.4 The Standard Model Lagrange Density

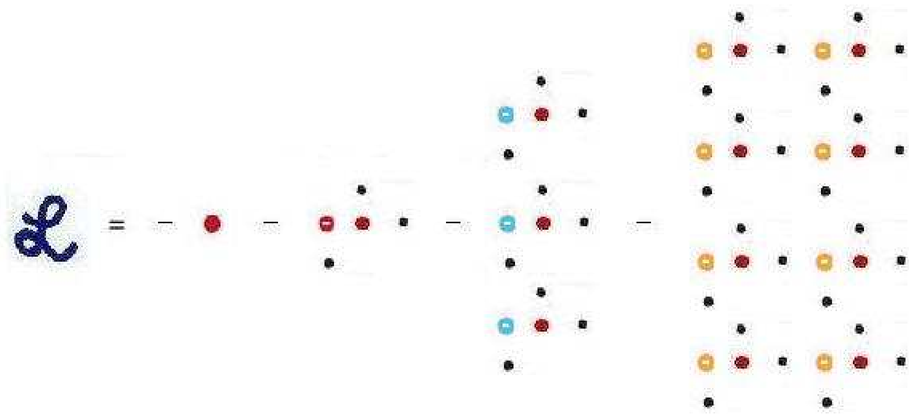
Describes all interactions of all subatomic forces in a volume.

$$\mathfrak{L}_{\mathrm{SM}} = \bar{\psi} \gamma^{\mu} D_{\mu} \psi$$

$$D_{\mu} = \partial_{\mu} - i g_{\mathrm{EM}} Y A_{\mu} - i g_{\mathrm{weak}} \frac{\tau^a}{2} W^a_{\mu} - i g_{\mathrm{strong}} \frac{\lambda^b}{2} G^b_{\mu}$$

- γ^{μ} Spinor matrix (no details provided here).

- $g_{...}$ Coupling constant to force.
- Y Generator of U(1) symmetry.
- $\tau^{a(1-3)}$ Generator of SU(2) symmetry.
- $\lambda^{b(1-8)}$ Generator of SU(3) symmetry.
- A_μ, W^a_μ, G^b_μ Complex-valued 4-potentials, two with internal symmetries.



4.11.5 Defining the Multiplication Operator

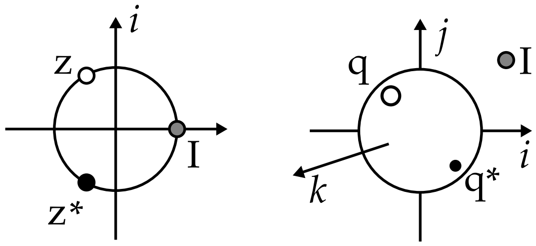
Given a pair of complex-valued 4-vectors,
need to generate a real scalar.
Four components:

1. $(a, bi)^* = (a, -bi)$ Complex conjugation.
2. $(\phi, \vec{A})^p = (\phi, -\vec{A})$ Parity operator.
3. $g_{\mu\nu}$ Metric tensor.
4. $\frac{A^\mu}{|A|}$ Potentials normalized to themselves.

Define multiplication of 4-potentials in the standard model as:

$$\frac{A^\mu}{|A|}\frac{A^{\nu*p}}{|A|}g_{\mu\nu} = \frac{g_{tt}|A_t|^2 - g_{xx}|A_x|^2 - g_{yy}|A_y|^2 - g_{zz}|A_z|^2 - g_{\mu\nu}|A^\mu A^\nu|_{\mu\neq\nu}}{|A|^2}$$

U(1)xSU(2)



Conjugation Parity

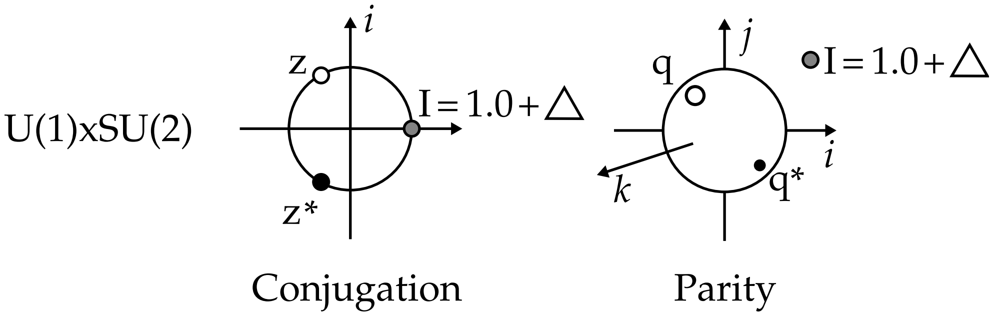
4.11.6 Multiplication Operator in Spacetime

- $\frac{A^{\mu}}{|A|}\frac{A^{\nu*p}}{|A|}g_{\mu\nu} = 1.0$ In flat spacetime.
- $\frac{A^{\mu}}{|A|}\frac{A^{\nu*p}}{|A|}g_{\mu\nu} = 1.0 + \delta$ In curved spacetime.

In curved spacetime, mass breaks U(1), SU(2), and SU(3) symmetry in a precise way (circles get larger).

Y, τ^a, λ^b and the Higgs particle are not needed.

No new symmetry was added to the standard model. No new particle can be added. Instead, it may turn out that every particle can "act like a graviton" when it is involved with a distance measurement of the field.



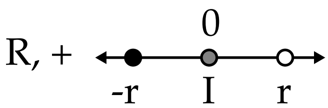
4.11.7 Summary: The Standard Model

Math:

$$\frac{A^{\mu}}{|A|}\frac{A^{\nu*p}}{|A|}g_{\mu\nu} = \frac{g_{tt}|A_t|^2 - g_{xx}|A_x|^2 - g_{yy}|A_y|^2 - g_{zz}|A_z|^2 - g_{\mu\nu}|A^{\mu}A^{\nu}|_{\mu\neq\nu}}{|A|^2}$$

Pictures:

1



U(1)xSU(2)xSU(3)

3

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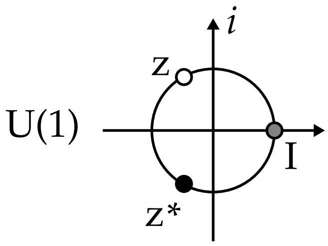

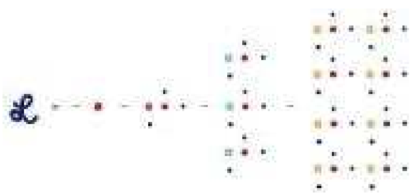

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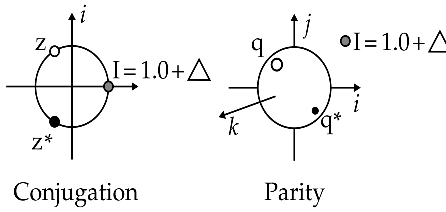


5

4.12 Must Do Physics Done

1. $F_g = -Gm\psi\hat{R}$ Like charges attract.
2. $+m$ One charge.
3. $\rho = \nabla^2\phi$ Newton's gravitational field equation.
4. $m\frac{d^2\vec{R}}{dt^2} = -\frac{GMm}{R^2}\hat{R}$ Newton's law of gravity under classical conditions.
5. $d\tau^2 = (1 - 2\frac{GM}{c^2R} + 2(\frac{GM}{c^2R})^2)\,dt^2 - (1 + 2\frac{GM}{c^2R})\,\frac{dR^2}{c^2}$
   Consistent with the Schwarzschild metric.
6. $F_{\text{EM}} = q\,\vec{E}$ Like charges repel.
7. $\pm q$ Two distinct charges.
8. $\rho = \vec{\nabla}\cdot\vec{E} \quad \vec{J} = -\frac{\partial\vec{E}}{c\,\partial t} + \vec{\nabla}\times\vec{B}$ Maxwell source equations.
9. $0 = \vec{\nabla}\cdot\vec{B} \quad \vec{0} = \frac{\partial\vec{B}}{c\partial t} + \vec{\nabla}\times\vec{E}$ Maxwell homogeneous equations.
10. $F^\mu = q\,\frac{U_\mu}{c}(A^{\mu,\nu} - A^{\nu,\mu})$ Lorentz force.
11. Unified field emission modes can be quantized.
12. Works with the standard model.
13. Indicates origin of mass.
14. LIGO (gravity wave polarization).
15. Rotation profiles of spiral galaxies.
16. Big Bang constant velocity distribution.

Caveats:
5. Check metric derivation. Proposal can be confirm/rejected by experiment.
15. Actual, detailed calculations must be compared with data.
16. See 15.

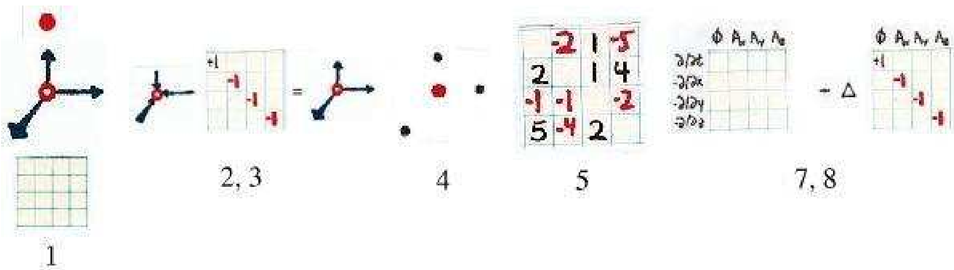
Appendices: Tensors and Units

4.13 Tensors

Scalars, vectors, and matrices are tensors.

Some new words are needed to generalize their properties.

1. Simple tensors
2. Covariant versus contravariant.
3. Going from covariant to contravariant.
4. Einstein's summation convention.
5. Symmetric versus antisymmetric tensors.
6. Derivatives in flat spacetime.
7. Covariant derivatives in curved spacetime.



4.13.1 Simple Tensors

Useful no matter the coordinate system or dimension.

The simple tensors:

- Rank-0 tensor, a scalar.
- Rank-1 tensor, a vector.
- Rank-2 tensor, a matrix.

For these lectures, only 4-vectors and 4x4 matrices are used.

4.13.2 Covariant versus Contravariant

Subscript versus superscript.

- $A_\mu = (A_0, -A_1, -A_2, -A_3) = (\phi, -\vec{A})$ **Co**variant potential vector.
- $A^\mu = (A_0, A_1, A_2, A_3) = (\phi, \vec{A})$ **Contra**variant potential vector.
- μ, ν, ϖ Greek indices go from 0, 1, 2, 3.
- u, v Roman indices go from 1, 2, 3.

Memory aid: co is a commie, commies are low, negative; contras are proud, positive, up against a wall.

4.13.3 Going from Covariant to Contravariant

Use the rank-2 metric tensor, $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ [flat spacetime].

- $g_{\mu\nu} A^\nu = A_\mu$ Lower an index.
- $g^{\mu\varpi} g^{\nu\sigma} A_{\varpi\sigma} = A^{\mu\nu}$ Raise two indices.

Memory aid: If the metric g has two indices raised up to the sky, it will be raising an index.

4.13.4 Einstein's Summation Convention

Contract same co- contra- index,
No $\sum$ needed.

- $A^\mu A_\mu = \phi^2 - \vec{A} \cdot \vec{A}$ Rank-0 tensor result.

- $A^{\mu}A^{\nu}g_{\mu\nu} = A^{\mu}A_{\mu}$ Metric contracts two contravariant vectors.
- $A^{\mu}A_{\mu} = A^{\nu}A_{\nu}$ Dummy variable names.

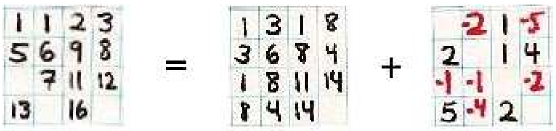
4.13.5 Symmetric versus Antisymmetric Tensors

Swap indices, see if sign does/does not flip for all.

- $A^{\mu\nu} = A^{\nu\mu}$ Symmetric, all keep sign.
- $A^{\mu\nu} = -A^{\nu\mu}$ Antisymmetric, all flip signs.
- $A^{\mu\nu} \neq A^{\nu\mu}$ Asymmetric, no pattern.

Any asymmetric tensor can be represented by a symmetric tensor (averaged values of 2 indices) and an antisymmetric tensor (+ and - deviations from average).

$A^{\mu\nu} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}) + \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$



Average Joe & The Deviants

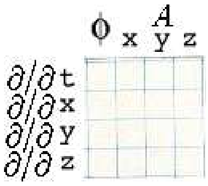
4.13.6 Derivatives in Flat, Euclidean Spacetime

4-derivatives: time and 3-space derivatives in a rank-1 tensor.

Signs of covariant and contravariant derivatives flip (arg!).

- $\partial_{\mu} = (\frac{\partial}{\partial t}, \vec{\nabla})$ Covariant derivative.
- $\partial^{\mu} = (\frac{\partial}{\partial t}, -\vec{\nabla})$ Contravariant derivative.
- $\partial^{\mu}A^{\nu} = A^{\nu,\mu}$ The comma convention.
- $\partial_{\mu}\partial^{\mu} = \Box^2 = (\frac{\partial^2}{\partial t^2} - \nabla^2)$ The D'Alembertian operator.

Alternate representation: The asymmetric rank-2 tensor that results from taking the 4-derivative of a 4-vector can be represented by the symmetric average amount of change tensor plus the antisymmetric deviation of change tensor.



4.13.7 Covariant Derivatives in Curved Spacetime

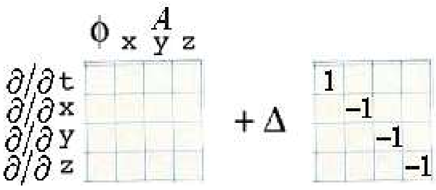
Covariant derivative = normal derivative $\pm$ derivative of the metric.

The connection or Christoffel symbol ($\Gamma_{\varpi}{}^{\mu\nu}$) handles the derivatives of the metric.

";" the semicolon convention for covariant derivatives.

- $\nabla^{\mu} A^{\nu} = \partial^{\mu} A^{\nu} - \Gamma_{\varpi}{}^{\mu\nu} A^{\varpi}$ Derivative = normal - change in metric.
- $\nabla^{\mu} A_{\nu} = \partial^{\mu} A_{\nu} + \Gamma^{\mu}{}_{\nu\varpi} A^{\varpi}$ Covariant derivative of a contravariant vector.
- $\nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ Independent of the metric because $\Gamma^{\varpi}{}_{\mu\nu} = \Gamma^{\varpi}{}_{\nu\mu}$.
- $\nabla^{\mu} A^{\nu} + \nabla^{\nu} A^{\mu} = \partial^{\mu} A^{\nu} + \partial^{\nu} A^{\mu} - 2\Gamma_{\varpi}{}^{\mu\nu} A^{\varpi}$.

Omission: The details of Christoffel symbol are not discussed here.

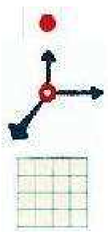


4.13.8 Summary: Tensors

Math:

$A^{\mu\nu} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}) + \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$

Pictures:



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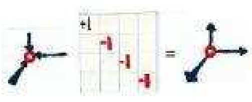

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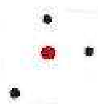

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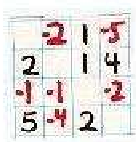

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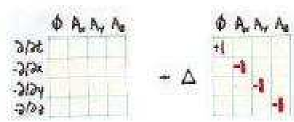

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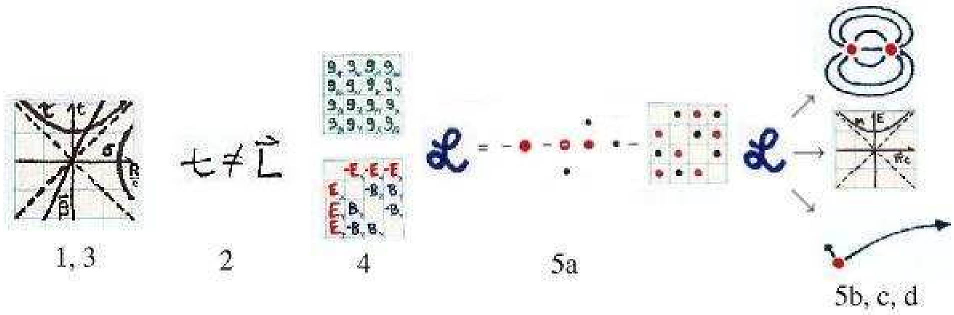
4.14 Units

1. Basic units.
2. Units for conversion factors.
3. Units for spacetime.
4. Units for potentials, fields, & charges.
5. Units in action:
    a) Lagrange densities.
    b) Euler-Lagrange equations (fields).
    c) Momentum.
    d) Force.



4.14.1 Basic Units

- t Time.
- L Length.
- m Mass.

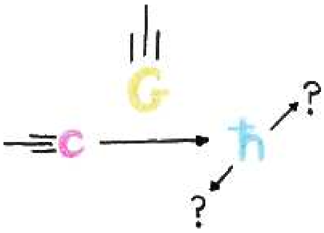
For EM, Gaussian units will be used. Units of electric charge are found from Coulomb's law: $F = \frac{qq'}{R^2} \rightsquigarrow \frac{mL}{t^2}$ so $q \rightsquigarrow \frac{\sqrt{mL^3}}{t}$ where " $\rightsquigarrow$ " means "has units of".

Spacetime **Energy/Momentum**

Units for Conversion Factors
For gravity, spacetime, & quantum mechanics.

- $G \rightsquigarrow \frac{L^3}{mt^2}$ Gravitational constant.
- $c \rightsquigarrow \frac{L}{t}$ Speed of light.

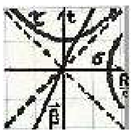
- $h \rightsquigarrow \frac{mL^2}{t}$ Planck's constant.



4.14.2 Units for Spacetime

Where all events of gravity, EM, and quantum mechanics take place.

- $V \rightsquigarrow L^3$ Volume.
- $\tau^2 = t^2 - \frac{\vec{R} \cdot \vec{R}}{c^2} \rightsquigarrow t^2$ Interval squared.
- $\sigma^2 = \vec{R} \cdot \vec{R} - c^2 t^2 = -c^2 \tau^2 \rightsquigarrow L^2$ 4D-distance squared.
- $\gamma = \frac{1}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}} = \frac{\partial t}{\partial \tau} \rightsquigarrow -$ Stretch factor.
- $\vec{\beta} = \frac{\vec{v}}{c} \rightsquigarrow -$ Relativistic 3-velocity
- $U^\mu = (c\frac{\partial t}{\partial \tau}, \frac{\partial \vec{R}}{\partial \tau}) = (c\gamma, c\gamma\vec{\beta}) = (\frac{E}{mc}, \frac{\vec{\pi}}{mc}) \rightsquigarrow \frac{L}{t}$ Velocity vector.



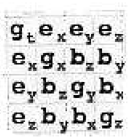
Spacetime

4.14.3 Units for Potentials, Fields, & Charges

The way to describe where stuff is everywhere, everywhen.

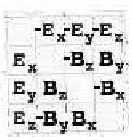
- $A^\mu = (\phi, \vec{A}) \rightsquigarrow \frac{\sqrt{m}}{\sqrt{L}}$ Potential vector.
- $A^{\mu;\nu} \rightsquigarrow \vec{g} \rightsquigarrow \vec{E} \rightsquigarrow \vec{B} \rightsquigarrow \frac{\sqrt{m}}{t\sqrt{L}}$ Derivatives of potential vectors (fields!).
- $q \rightsquigarrow \frac{\sqrt{mL^3}}{t} \rightsquigarrow \sqrt{G}\, m\, [\frac{\sqrt{L^3}}{t\sqrt{m}} m] \rightsquigarrow \sqrt{hc}\, [\frac{L\sqrt{m}}{\sqrt{t}} \frac{\sqrt{L}}{\sqrt{t}}]$ Charge.
- $J^\mu = \frac{q}{V}\frac{U^\mu}{\gamma c} = (\frac{q}{V}, \frac{q}{V}\vec{\beta}) \rightsquigarrow \frac{\sqrt{m}}{t\sqrt{L^3}} \rightsquigarrow \frac{\sqrt{G}\, m}{V} [\frac{\sqrt{L^3}}{t\sqrt{m}} m \frac{1}{L^3}] \rightsquigarrow \frac{\sqrt{hc}}{V} [\frac{L\sqrt{m}}{\sqrt{t}} \frac{\sqrt{L}}{\sqrt{t}} \frac{1}{L^3}]$
  Current density vector.

Gravity



g_t	e_x	e_y	e_z
e_x	g_x	b_z	b_y
e_y	b_z	g_y	b_x
e_z	b_y	b_x	g_z

EM

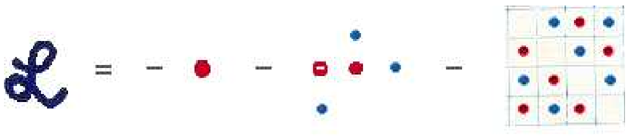


	$-E_x$	$-E_y$	$-E_z$
E_x		$-B_z$	B_y
E_y	B_z		$-B_x$
E_z	$-B_y$	B_x	

4.14.4 Units in Action: Lagrange Density

Lagrange Density, where all mass, energy, and interactions are in a volume.

- $\mathfrak{L} \rightsquigarrow \frac{m}{L^3}$ Mass density.
- $\mathfrak{L} \rightsquigarrow \frac{m}{\gamma V}\,[\frac{m}{L^3}] \rightsquigarrow \frac{1}{c^2}\,\frac{q}{V}\,\frac{U^\mu}{\gamma}\,A_\mu\,[\frac{t^2}{L^2}\,\frac{\sqrt{m\,L^3}}{t}\,\frac{1}{L^3}\,\frac{L}{t}\,\frac{\sqrt{m}}{\sqrt{L}}] \rightsquigarrow$
  $\frac{\sqrt{G}}{c^2}\,\frac{m}{V}\,\frac{U^\mu}{\gamma}A_\mu\,[\frac{\sqrt{L^3}}{t\sqrt{m}}\,\frac{t^2}{L^2}\,m\,\frac{1}{L^3}\,\frac{L}{t}\,\frac{\sqrt{m}}{\sqrt{L}}] \rightsquigarrow \frac{1}{c^2}\,A^{\mu;\nu}A_{\mu;\nu}\,[\frac{t^2}{L^2}\,\frac{\sqrt{m}}{t\sqrt{L}}\,\frac{\sqrt{m}}{t\sqrt{L}}]$
  Equivalent units.

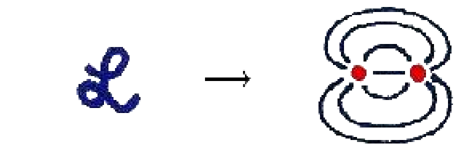


Lagrangian of E M

4.14.5 Units in Action: Euler-Lagrange Equations

Euler-Lagrange equations, generates field equations given a Lagrange density.

- $c\,\frac{\partial \mathfrak{L}}{\partial \phi} = c\,\partial_\mu \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi)}$ From principle of least action.
- $c\frac{\partial \mathfrak{L}}{\partial \phi}\,[\frac{L}{t}\,\frac{m}{L^3}\,\frac{\sqrt{L}}{\sqrt{m}}] \rightsquigarrow \frac{q}{V}\,[\frac{\sqrt{m\,L^3}}{t}\,\frac{1}{L^3}] \rightsquigarrow \nabla A^{\mu;\nu}\,[\frac{1}{L}\,\frac{\sqrt{m}}{t\sqrt{L}}]$
  $\rightsquigarrow \nabla \vec{g}\,[\frac{1}{L}\,\frac{\sqrt{m}}{t\sqrt{L}}] \rightsquigarrow \nabla \vec{E}\,[\frac{1}{L}\,\frac{\sqrt{m}}{t\sqrt{L}}] \rightsquigarrow \nabla \vec{B}\,[\frac{1}{L}\,\frac{\sqrt{m}}{t\sqrt{L}}] \rightsquigarrow J^\mu[\frac{\sqrt{m}}{t\sqrt{L^3}}]$ Equivalent units.



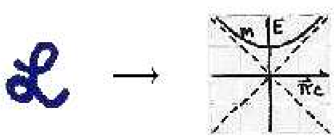
Lagrangian to fields

4.14.6 Units in Action: Momentum

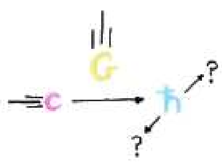
Energy and 3-momentum from a derivative of a Lagrange density.

- $\pi^\mu = h\sqrt{G}\frac{\partial \mathfrak{L}}{\partial \frac{\partial A^\mu}{c\,\partial t}} \rightsquigarrow \frac{m\,L^2}{t^2}$ Derivative of the Lagrange density.
- $\pi^\mu\,[\frac{m\,L^2}{t^2}] \rightsquigarrow h\sqrt{G}\frac{\partial \mathfrak{L}}{\partial \frac{\partial A^\mu}{c\,\partial t}}\,[\frac{m\,L^2}{t}\,\frac{\sqrt{L^3}}{t\sqrt{m}}\,\frac{L}{t}\,\frac{t\sqrt{L}}{\sqrt{m}}\,\frac{m}{L^3}]$ Equivalent units.

Note: units suggest relativistic (c), quantum (h) gravity (G).



Lagrangian to Energy/Mo



Units of Relativistic (c), quantum (h), Gravity (G)

4.14.7 Units in Action: Relativistic Force

- $F^{\mu} = \text{cause} = \frac{\partial m U^{\mu}}{\partial \tau}$ Force is a cause which has an effect on momentum.
- $F^{\mu}\,_{[\frac{m\,L}{t^2}]} \rightsquigarrow \frac{1}{c}\, q U_{\nu} \nabla^{\mu} A^{\nu}\,_{[\frac{t}{L} \frac{\sqrt{m\,L^3}}{t} \frac{L}{t} \frac{\sqrt{m}}{t\sqrt{L}}]} \rightsquigarrow \frac{\sqrt{h}}{\sqrt{c}}\, n U_{\nu} \nabla^{\mu} A^{\nu}\,_{[\frac{L\sqrt{m}}{\sqrt{t}} \frac{\sqrt{t}}{\sqrt{L}} \frac{L}{t} \frac{\sqrt{m}}{t\sqrt{L}}]}$

  $\rightsquigarrow \frac{\sqrt{G}}{c}\, m U_{\nu} \nabla^{\mu} A^{\nu}{}_{[\frac{\sqrt{L^3}}{t\sqrt{m}} \frac{t}{L}\, m\, \frac{L}{t} \frac{\sqrt{m}}{t\sqrt{L}}]}$

  Equivalent units.

Lagrangian to Force

4.14.8 Summary: Units

Math:

$q \rightsquigarrow \frac{\sqrt{m\,L^3}}{t} \rightsquigarrow \sqrt{G}\, m\,_{[\frac{\sqrt{L^3}}{t\sqrt{m}}\, m]} \rightsquigarrow \sqrt{h\,c}\,_{[\frac{L\sqrt{m}}{\sqrt{t}} \frac{\sqrt{L}}{\sqrt{t}}]}$

Pictures: