There is no place like home: Looking for a metric equation for gravity within the structure of the Maxwell equations

Douglas B. Sweetser

September 30, 2001

1340 Commonwealth Ave. Apt. 7, Allston, MA 02134

September 30, 2001

Abstract

The Maxwell equations written in the Lorenz gauge are known, at least mathematically, to have four modes of transmission: two transverse modes for electrodynamics, a longitudinal, and a scalar mode. The probabilities of the last two modes cancel each other out for photons in a vacuum, but that does not have to be the case for a nonhomogeneous equation. One scalar potential solution to the equations of motion is found, the inverse of an interval between two events squared. The force field created by the potential is constructed by comparison with the classical Newtonian field. The Lagrangian $L = -J^{\mu}A_{\mu} - \frac{1}{2}(\partial^{\mu}A^{\nu})(\partial_{\mu}A_{\nu})$ can contribute to the scalar mode, but still forms the Maxwell equations in the Lorenz gauge. A relativistic force equation is proposed, created by the product of charge, normalized force field, and 4-velocity: $\frac{\partial mU^{\mu}}{\partial\tau} = kq\frac{\partial^{\mu}A^{\nu}}{|A|}U_{\nu}$. The solution to the force equation using the inverse square interval potential is found. Eliminating the constants generates a metric equation, $(\partial\tau)^2 = e^{-2\frac{GM}{c^2\tau}}(\partial t)^2 - e^{2\frac{GM}{c^2\tau}}(\partial\overrightarrow{R})^2$, where τ is a lightlike interval with almost the same magnitude as the radius R of separation between source and test masses. For a weak gravitational field, the metric will pass the same tests as the Schwarzschild metric of general relativity. The two metrics differ for higher order terms, which makes the proposed metric distinct and testable experimentally. A constant-velocity solution exists for the gravitational force equation for a system with an exponentially-decaying mass distribution. The dark matter hypothesis is not needed to explain the constant-velocity profiles seen for some galaxies. The proposal may also have implications for classical big bang theory.

1 An opportunity for gravity within the Maxwell equations

The Maxwell equations can be quantized in a manifestly covariant form by fixing the gauge.[5] The starting point is the 4-potential A^{μ}. There are four modes of transmission for photons corresponding to the four degrees of freedom: two transverse, one scalar, and one longitudinal. Gupta calculated that for photons in a vacuum, the probability of the scalar mode cancels that of the longitudinal mode, so both are virtual. He notes that this does not alway have to be the case for the nonhomogeneous Maxwell equations, which is the focus of this work.

My hypothesis is that a dominant scalar mode for the Maxwell equations in the Lorenz gauge is gravity. The hypothesis makes several predictions even at this preliminary stage. First, the math of gravity and electromagnetism should be similar but not identical. The inverse square form of Newton's law of gravity was a direct inspiration for Coulomb's law. Gravity should be more symmetric than electromagnetism because the mode is scalar, instead of transverse. The second rank field strength tensor in general relativity is symmetric while the analogous tensor for the electromagnetic field is antisymmetric. Since the mode of gravity is orthogonal to electromagnetism, the charges can be likewise, so there will be no simple relationship between gravitational charge (mass) and electric charge. Gravitational waves in general relativity are transverse, so this proposal is distinct from general relativity. Nature exploits all the math available, so it is unreasonable to suppose that the scalar mode is never used for anything. Whatever phenomenon exploits the scalar mode must be similar, but just as important as electromagnetism. Gravity is a natural candidate.

An algebraic road will be constructed starting from a solution to the Maxwell equations in the Lorenz gauge to a curved metric. Many of the steps will be justified by the need to be consistent with Newton's law of gravity in the classical limit. The reward of this work is a metric which is similar enough to the Schwarzschild metric of general relativity to agree with all the experimental results to post-Newtonian accuracy, yet differs for higher order terms. Thus the proposal can be confirmed or refuted by more precise tests of the metric. The second major prediction concerns the velocity profile of spiral galaxies. For a mass distribution that decays exponentially, the equations of motion for the scalar mode predict a stable and flat velocity profile with increasing radius. This eliminates the need for the dark matter hypothesis.

2 A gravitational field inside Maxwell

Newton's classical gravitational law arises from a scalar potential. Here is the scalar field equation:

$$\nabla^2 \phi = 4\pi G \rho. \tag{1}$$

For the case of a vacuum, when $\rho = 0$, this is known as the Laplace equation. For a spherically symmetric source, one solution is:

$$\phi = -\frac{GM}{\sqrt{x^2 + y^2 + z^2}}. \tag{2}$$

The problem with the field equation is that the Laplace operator does not have a time differential. Any change in in the mass density propagates at infinite speed, in conflict with special relativity.[10, Chapter 7] One way to derive the field equations of general relativity involves making Newton's law of gravity consistent with the finite speed of light.[9]

A way to repair the field equations is to use the D'Alembertian operator, which is four dimensional. That expression is identical to the A^0 component of the Maxwell equations in the Lorenz gauge:

$$\Box^2 A^\mu = 4\pi k J^\mu. \tag{3}$$

If one is studying scalar (or possibly longitudinal) modes, J^μ is the mass density. If one is working with transverse modes, J^μ is the electric charge density. Since the modes are orthogonal, the sources can be also.

To be consistent with the classic scalar potential yet still be relativistic, the potential must have x^2, y^2, z^2, and t^2. This suggests a particular solution to the field equations (Eq. 3):

$$A^\mu = (\frac{1}{c^2 t^2 - x^2 - y^2 - z^2}, 0, 0, 0). \tag{4}$$

This potential is interesting for several reasons. It is the inverse of the Lorentz-invariant interval squared. Like mass, the 4-potential will not be altered by a change in an inertial reference frame. The interval between any two events will contribute to the potential. General relativity applies to any form of energy, including gravitational field energy. A potential that embraces every interval may have a broad enough scope to do the work of gravity.

The potential also has serious problems. Classical gravity depends on an inverse square force field, not an inverse square potential. Taking the derivative of the potential puts a forth power of the interval in the denominator. At this point, I could stop and say that this potential has nothing to do with gravity because it has the wrong dependence on distance. An alternative is to look for an algebraic way to repair the problem. This is the type of approach used by the early workers in quantum mechanics like de Broglie, and will be adopted here. The equations of motion (Eq. 3) can be normalized to the magnitude of the 4-potential:

$$\frac{\Box^2 A^\mu}{|A|} = 4\pi k J^\mu. \tag{5}$$

Since the magnitude of the potential is the inverse interval squared, the resulting equation has only an interval squared in the denominator. An interval is not necessarily the same as the distance R between the source and test mass used in the classical theory. However, I can impose a selection rule that in the classical limit, the only events that contribute to the potential are those that are timelike separated between the source and the test masses. It takes a timelike interval to know that the source is a distance R away. Action-at-a-distance respects the speed of light as it must.

3 Search for the source mass

Where is the source mass in the potential? All that has been discussed so far is an interval, a distance, nothing about mass. An idea from general relativity will be borrowed, that mass can be treated geometrically if multiplied by the constants $\frac{G}{c^2}$. The distance between the Earth and the Sun is approximately $1.5x10^{11}m$, while the Sun's mass expressed in units of distance, $\frac{GM_{Sun}}{c^2}$, is $1.5x10^3m$, eight orders of magnitude smaller. The overall length of the interval will not be changed noticeably if the spatial separation and the Sun's mass expressed as a distance are summed. However, the force field is the derivative of the potential, and any change in position in spacetime will have a far greater effect proportionally on the smaller geometric mass than the spatial separation. Make the following change of variables:

$$\begin{aligned} t &\rightarrow t' = A + \frac{GM}{2c^2 A} t \\ \vec{R} &\rightarrow \vec{R}' = \vec{B} + \frac{GM}{2c^2|\vec{B}|}\vec{R}, \end{aligned} \tag{6}$$

where A and $\vec{B}$ are locally constants such that $\tau^2 \cong A^2 - \vec{B}^2$. The change of variables is valid locally, but not globally, since it breaks down for arbitrarily long time or distance away. General relativity is also valid locally and not globally. The derivative of the normalized interval squared is approximately:

$$\begin{aligned} \frac{1}{|\frac{1}{\tau^2}|}\frac{\partial \frac{1}{\tau^2}}{\partial t} &\cong -\frac{GM}{c^2\tau^2} \\ \frac{1}{|\frac{1}{\tau^2}|}\vec{\nabla}\frac{1}{\tau^2} &\cong \frac{GM}{c^2\tau^2}\widehat{R}. \end{aligned} \tag{7}$$

This should look familiar, remembering that the magnitude of τ^2 is the same as R^2, differing only by the geometric mass of the source.

4 A Lagrangian for four modes

Gupta wanted to quantize the Maxwell equations using a form that was manifestly covariant in its explicit treatment of time and space. He fixed the gauge with this Lagrangian:

$$L = -J^{\mu} A_{\mu} - \frac{1}{2}(\partial^{\mu} A_{\mu})^2 - \frac{1}{4}(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}). \qquad (8)$$

The equations of motion for this Lagrangian are the Maxwell equations in the Lorenz gauge (Eq. 3). The problem with the Lagrangian is that the field strength tensor is antisymmetric. Due to the zeros along the diagonal, it cannot contribute directly to a scalar mode. What is needed is a Lagrangian that could contribute directly to the scalar mode but still have the same equation of motion. Here is such a Lagrangian:

$$L = -J^{\mu} A_{\mu} - \frac{1}{2}(\partial^{\mu} A^{\nu})(\partial_{\mu} A_{\nu}). \qquad (9)$$

This is not as miraculous as it might first appear. It is the first of four terms generated in the contraction of the electromagnetic field strength tensor. In essence, I have chosen not to discard information, which is what happens in making the field strength tensor antisymmetric. The one remaining modification is to normalize both the Lagrangian and equations of motion to the size of the potential.

5 From a 4-force to a metric

A relativistic 4-force is the change in momentum with respect to the interval. The covariant force law is similar in form to the one for electromagnetism except that the second rank tensor is asymmetric and normalized:

$$F = \frac{\partial p}{\partial \tau} = m \frac{\partial U^{\mu}}{\partial \tau} + U^{\mu} \frac{\partial m}{\partial \tau} = kq \frac{\partial^{\mu} A^{\nu}}{|A|} U_{\nu}. \qquad (10)$$

In the first application of the force law, assume the derivative of the mass with respect to the interval is zero. For the scalar mode, assume the charge q is the gravitational test mass. Experiments have demonstrated that gravitational and inertial masses are equal.[15] The inverse interval squared potential leads to the following equation of motion:

$$\left(\frac{\partial^2 t}{\partial \tau^2} + \frac{GM}{c^2 \tau^2} \frac{\partial t}{\partial \tau}, \frac{\partial^2 \vec{R}}{\partial \tau^2} - \frac{GM}{c^2 \tau^2} \frac{\partial \vec{R}}{\partial \tau}\right) = (0, \vec{0}). \qquad (11)$$

Solve this second-order differential equation for the spacetime position:

$$
\begin{aligned}
t &= c_1(\tau e^{\frac{GM}{c^2\tau}} - \frac{GM}{c^2}Ei(\frac{GM}{c^2\tau})) + c_2 \\
\overrightarrow{R} &= \overrightarrow{C}_1(\tau e^{-\frac{GM}{c^2\tau}} + \frac{GM}{c^2}Ei(-\frac{GM}{c^2\tau})) + \overrightarrow{C}_2,
\end{aligned} \tag{12}
$$

where Ei is the exponential integral, $Ei(t) = \int_{-\infty}^{t} \frac{e^t}{t} dt$. The exponential integral plays a role in quantum mechanics, so its presence is interesting.

Eight constants need to be eliminated: $(c_1, \overrightarrow{C_1})$ and $(c_2, \overrightarrow{C}_2)$. Take the derivative of the spacetime position with respect to τ. This eliminates four constants, $(c_2, \overrightarrow{C}_2)$. The result is a 4-velocity:

$$
\begin{aligned}
\frac{\partial t}{\partial \tau} &= c_1 e^{\frac{GM}{c^2\tau}} \\
\frac{\partial \overrightarrow{R}}{\partial \tau} &= \overrightarrow{C}_1 e^{-\frac{GM}{c^2\tau}}.
\end{aligned} \tag{13}
$$

In flat spacetime, $U_\mu U^\mu = 1$, providing four more constraints. Spacetime is flat if $M \to 0$ or $\tau \to \infty$, leading to $e^{\pm\frac{GM}{c^2\tau}} \to 1$:

$$
(\frac{\partial t}{\partial \tau})^2 - (\frac{\partial \overrightarrow{R}}{\partial \tau}) \cdot (\frac{\partial \overrightarrow{R}}{\partial \tau}) = c_1^2 - \overrightarrow{C}_1 \cdot \overrightarrow{C}_1 = 1. \tag{14}
$$

Solve for c_1^2 and $\overrightarrow{C}_1 \cdot \overrightarrow{C}_1$:

$$
\begin{aligned}
c_1^2 &= e^{-2\frac{GM}{c^2\tau}}(\frac{\partial t}{\partial \tau})^2 \\
\overrightarrow{C}_1 \cdot \overrightarrow{C}_1 &= e^{2\frac{GM}{c^2\tau}}(\frac{\partial \overrightarrow{R}}{\partial \tau}) \cdot (\frac{\partial \overrightarrow{R}}{\partial \tau}).
\end{aligned} \tag{15}
$$

Substitute back into the flat spacetime constraint. Rearrange into a metric:

$$
(\partial \tau)^2 = e^{-2\frac{GM}{c^2\tau}}(\partial t)^2 - e^{2\frac{GM}{c^2\tau}}(\partial \overrightarrow{R})^2. \tag{16}
$$

As expected, this becomes the Minkowski metric for flat spacetime if $M \to 0$ or $\tau \to \infty$. For a weak field, write the Taylor series expansion in terms of the source mass over the interval to second-order in $\frac{GM}{c^2\tau}$:

$$
\begin{aligned}
&\partial \tau^2 = (1 - 2\frac{GM}{c^2\tau} + 2(\frac{GM}{c^2\tau})^2)\partial t^2 - \\
&-(1 + 2(\frac{GM}{c^2\tau}) + 2(\frac{GM}{c^2\tau})^2)\partial \overrightarrow{R}^2 + O((\frac{GM}{c^2\tau})^3).
\end{aligned} \tag{17}
$$

Contrast this with the Schwarzschild solution in isotropic coordinates expanded to second order in $\frac{GM}{c^2 R}$[10, Eq. 31.22]:

$$\partial\tau^2 = (1 - 2\frac{GM}{c^2 R} + 2(\frac{GM}{c^2 R})^2)\partial t^2 -$$

$$-(1 + 2(\frac{GM}{c^2 R}) + 2.5(\frac{GM}{c^2 R})^2)\partial \vec{R}^2 + O((\frac{GM}{c^2 R})^3). \tag{18}$$

The magnitude of the lightlike interval τ in Eq. 17 is nearly identical to the radius R in the Schwarzschild metric, the difference being the geometric mass of the source included in the interval τ. The metric for the scalar potential will pass the same weak field tests of general relativity as the Schwarzschild metric to post-Newtonian accuracy, which does not use the second order spatial term.[15] The difference in the higher order terms can be the basis of an experimental test to distinguish this proposal from general relativity. Since the effect is second order in the field term, such a test will challenge experimental techniques.

The two metrics are numerically very similar for weak fields, but mathematically distinct. For example, the Schwarzschild metric is static, but the new metric contains a dependence on time, so is dynamic (but only locally, for small amounts of time). The Schwarzschild metric has a singularity at $R = 0$. The metric for the scalar mode becomes undefined for lightlike intervals. This might pose less of a conceptual problem, since light has no rest mass, and the transverse mode describes the motion of massless particles.

6 A constant velocity profile solution

There are two problems with a classical Newtonian gravity explanation of the flat velocity profiles of thin spiral galaxies with a mass distribution that decays exponentially. [3, 7, 8, 14] First, the galaxies should have a Keplarian decline in the velocity profile with distance.[11] Second, the a thin spiral galaxy is not stable a stable solution because a small disturbance should cause it to collapse.[12] The work on dark matter is an attempt to remedy these problems.

In the previous section, the system had a constant effective point-source mass with a velocity profile that decayed with distance. Here in an attempt to explain the spiral galaxies, the opposite situation is examined, where the velocity profile is a constant, but the mass distribution decays exponentially with distance. The force equation in this situation is:

$$U^\mu \frac{\partial m}{\partial \tau} = m \frac{\partial^\mu A^\nu}{|A|} U_\nu. \tag{19}$$

Gravity's effect is on the distribution of mass over spacetime where the velocity is constant. Make the same assumptions as used before. Presume a inverse interval squared potential. The interval τ has nearly the same magnitude as the distance between the source and test masses, except that it includes the source mass expressed as a distance. Assuming the equivalence principle this time does not lead to the cancellation of the test mass, but instead allows the test mass to be the focus of the following differential equation:

$$(\gamma(\frac{\partial m}{\partial \tau} + \frac{GM}{c^2 \tau^2} m), \gamma \vec{\beta} (\frac{\partial m}{\partial \tau} - \frac{GM}{c^2 \tau^2} m)) = (0, \vec{0}). \tag{20}$$

Solve for the mass flow:

$$(\gamma m, \gamma \vec{\beta} m) = (c e^{\frac{GM}{c^2 \tau}}, \vec{C} e^{-\frac{GM}{c^2 \tau}}). \tag{21}$$

The velocity is constant, so it is the test mass distribution that shows an exponential decay with respect to the interval, which is numerically almost the same as the radius. This is a stable solution. If the test mass keeps dropping of exponentially, the velocity profile will remain constant.

Look at the problem in reverse. The distribution of mass has an exponential decay with distance from the center. It must solve a differential equation with the velocity constant over that region of spacetime like the one proposed.

The exponential decay of the mass of a disk galaxy is only one solution to the gravitational force equation (Eq. 10). The behavior of larger systems, such as gravitational lensing caused by clusters, cannot be explained by the Newton's law.[1][4][13] It will remain to be seen if this proposal is sufficient to work on that scale.

7 Future directions

An algebraic path between a solution to the Maxwell equations in the Lorenz gauge and a metric gravitational theory has been shown. Like the early work in quantum mechanics, a collection of hunches is used to connect equations. One is left with the question of why this might work? Fortunately the answer is subtle enough that I did not have to mention my own area of study, four dimensional division algebras. The action of a gauge invariant theory cannot be inverted to generate the propagator needed for quantum mechanics.[6] Fixing the gauge makes the action invertible. This may appear to be a technical feature, but the author believes this is vital. If the operation of multiplication surpasses what can be done with division, then Nature cannot harness the most robust mathematical structure, a topological algebraic field, the foundation for doing calculus. Nature does calculus in four dimensions, and it is this requirement that fixes the gauge. In the future, when we understand how to do calculus with four dimensional automorphic functions, we may have a deep appreciation of Nature's methods.

For a spiral galaxy with an exponential mass distribution, dark matter is no longer needed to explain the flat velocity profile observed or the long term stability of such disks. Mass distributed over large distances of space has an effect on the mass distribution itself. This raises an interesting question: is there also an effect of mass distributed over large amounts of time? If the answer is yes, then this might solve two analogous riddles involving large time scales, flat velocity profiles and the stability of solutions. Classical big bang cosmology theory spans the largest time frame possible and faces two such issues. The horizon problem involves the extremely consistent velocity profile across parts of the Universe that are not casually linked.[10, p. 815] The flatness problem indicates how unstable the classical big bang theory is, requiring exceptional fine tuning to avoid collapse.[2] Considerable effort will be required to substantiate this tenuous hypothesis. Any insight into the origin of the unified engine driving the Universe of gravity and light is worthwhile.

References

[1] A. G. Bergmann, V. Petrosian, and R. Lynds. Gravitational lens images of arcs in clusters. *Astrophys. J.*, 350:23, 1990.

[2] R. H. Dicke and P. J. E. Peebles. *General relativity: An Einstein centenary survey.* Cambridge University Press, 1979.

[3] K. C. Freeman. On the disks of spiral and so galaxies. *Astrophys. J.*, 160:811–830, 1970.

[4] S. A. Grossman and R. Narayan. Gravitationally lensed images in abell 370. *Astrophys. J.*, 344:637–644, 1989.

[5] S. N. Gupta. Thoery of longitudinal photons in quantum electrodynamics. *Proc. Phys. Soc.*, 63:681–691, 1950.

[6] M. Kaku. *Quantum field theory: A modern introduction.* Oxford University Press, 1993.

[7] S. M. Kent. Dark matter in spiral galaxies. i. galaxies with optical rotation curves. *Astron. J.*, 91(6):1301–1327, 1986.

[8] S. M. Kent. Dark matter in spiral galaxies. ii. galaxies with h1 rotation curves. *Astron. J.*, 93(4):816–832, 1987.

[9] R. H. Kraichnan. Special-relativistic derivation of generally covariant gravitation theory. *Phys. Rev.*, 55:1118–1122, 1955.

[10] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* W. H. Freeman and Company, 1970.

[11] A. Toomre. On the distribution of matter within highly flattened galaxies. *Astrophys. J.*, 138(2):385–392, 1963.

[12] A. Toomre. On the gravitational stability of a disk of stars. *Astrophys. J.*, 139:1217, 1964.

[13] J. A. Tyson, F. Valdes, and R. A Wenk. Detection of systematic gravitational lens galaxy image alignments: Mapping dark matter in galaxy clusters. *Astrophys. J. Let.*, 349:L1, 1990.

[14] T. S. van Albada, J. N. Bahcall, and R. Sanscisi. Distribution of dark matter in the spiral galaxy ngc 3198. *Astrophys. J.*, 295(2):305–313, 1985.

[15] C. M. Will. *Theory and experiment in gravitational physics: Revised edition.* Cambridge University Press, 1993.