1 Newton's Second Law

The form of Newton's second law for three separate cases will be generated using quaternion operators acting on position quaternions. In classical mechanics, time and space are decoupled. One way that can be achieved algebraically is by having a time operator cat only on space, or by space operator only act on a scalar function. I call this the "2 zero" rule: if there are two zeros in the generator of a law in physics, the law is classical.

Newton's 2nd Law for an Inertial Reference Frame in Cartesian Coordinates

Define a position quaternion as a function of time.

$$R = \left(t, \vec{R}\right)$$

Operate on this once with the differential operator to get the velocity quaternion.

$$V = \left(\frac{d}{dt}, \vec{0}\right)\left(t, \vec{R}\right) = \left(1, \dot{\vec{R}}\right)$$

Operate on the velocity to get the classical inertial acceleration quaternion.

$$A = \left(\frac{d}{dt}, \vec{0}\right)\left(1, \dot{\vec{R}}\right) = \left(0, \ddot{\vec{R}}\right)$$

This is the standard form for acceleration in Newton's second law in an inertial reference frame. Because the reference frame is inertial, the first term is zero.

Newton's 2nd Law in Polar Coordinates for a Central Force in a Plane

Repeat this process, but this time start with polar coordinates.

$$R = (t, r\,\mathrm{Cos}[\theta], r\,\mathrm{Sin}[\theta], 0)$$

The velocity in a plane.

$$V = \left(\frac{d}{dt}, \vec{0}\right)(t, r\,\mathrm{Cos}[\theta], r\,\mathrm{Sin}[\theta], 0) =$$

$$= (1, \dot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\dot{\theta}, \dot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\dot{\theta}, 0)$$

Acceleration in a plane.

$$A = \left(\frac{d}{dt}, \vec{0}\right)(1, \dot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\dot{\theta}, \dot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\dot{\theta}, 0) =$$

$$= \Big(0, -2\dot{r}\,\mathrm{Sin}[\theta]\dot{\theta} - r\,\mathrm{Cos}[\theta]\left(\dot{\theta}\right)^2 + \ddot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\ddot{\theta},$$
$$2\dot{r}\,\mathrm{Cos}[\theta]\dot{\theta} - r\,\mathrm{Sin}[\theta]\left(\dot{\theta}\right)^2 + \ddot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\ddot{\theta}, 0\Big)$$

Not a pretty sight. For a central force, $\dot{\theta} = L/mr^2$, and $\ddot{\theta} = 0$. Make these substitution and rotate the quaternion to get rid of the theta dependence.

$$A = (\mathrm{Cos}[\theta], 0, 0, -\mathrm{Sin}[\theta])\left(\frac{d}{dt}, \vec{0}\right)^2(t, r\,\mathrm{Cos}[\theta], r\,\mathrm{Sin}[\theta], 0) =$$

$$= \left(0, \frac{L^2}{m^2 r^3} + \ddot{r}, \frac{2L\dot{r}}{m r^2}, 0\right)$$

The second term is the acceleration in the radial direction, the third is acceleration in the theta direction for a central force in polar coordinates.

Newton’s 2nd Law in a Noninertial, Rotating Frame

Consider the ”noninertial” case, with the frame rotating at an angular speed omega. The differential time operator is put into the first term of the quaternion, and the three directions for the angular speed are put in the next terms. This quaternion is then multiplied by the position quaternion to get the velocity in a rotating reference frame. Unlike the previous examples where t did not interfere with the calculations, this time it must be set explicitly to zero (I wonder what that means?).

$$V = \left(\frac{d}{dt}, \vec{\omega}\right)\left(0, \vec{R}\right) = \left(-\vec{\omega}.\vec{R}, \dot{\vec{R}} + \vec{\omega} x \vec{R}\right)$$

Operate on the velocity quaternion with the same operator.

$$A = \left(\frac{d}{dt}, \vec{\omega}\right)\left(-\vec{\omega}.\vec{R}, \dot{\vec{R}} + \vec{\omega} x \vec{R}\right) =$$

$$= \left(-\dot{\vec{\omega}}.\vec{R}, \ddot{\vec{R}} + 2\vec{\omega} x \dot{\vec{R}} + \dot{\vec{\omega}} x \vec{R} - \vec{\omega}.\vec{R}\vec{\omega}\right)$$

The first three terms of the 3-vector are the translational, coriolis, and azimuthal alterations respectively. The last term of the 3-vector may not look like the centrifugal force, but using a vector identity it can be rewritten:

$$-\vec{\omega}.\vec{R}\vec{\omega} = -\vec{\omega} x \left(\vec{\omega} x \vec{R}\right) + \left(\vec{\omega}\right)^2 \vec{R}$$

If the angular velocity an the radius are orthogonal, then

$$\vec{\omega} x \left(\vec{\omega} x \vec{R}\right) = \left(\vec{\omega}\right)^2 \vec{R} \text{ iff } \vec{\omega}.\vec{R} = 0$$

The scalar term is not zero. What this implies is not yet clear, but it may be related to the fact that the frame is not inertial.

Implications

Three forms of Newton’s second law were generated by choosing appropriate operator quaternions acting on position quaternions. The differential time operator was decoupled from any differential space operators. This may be viewed as an operational definition of ”classical” physics.

2 Oscillators and Waves

A professor of mine once said that everything in physics is a simple harmonic oscillator. Therefore it is necessary to get a handle on everything.

The Simple Harmonic Oscillator (SHO)

The differential equation for a simple harmonic oscillator in one dimension can be express with quaternion operators.

$$\left(\frac{d}{dt}, \vec{0}\right)^2 (0, x, 0, 0) + \left(0, \frac{k}{m}x, 0, 0\right) = \left(0, \frac{d^2 x}{dt^2} + \frac{k x}{m}, 0, 0\right) = 0$$

This equation can be solved directly.

$$x -> C[2] \, \text{Cos}\left[\frac{\sqrt{k}\, t}{\sqrt{m}}\right] + C[1] \, \text{Sin}\left[\frac{\sqrt{k}\, t}{\sqrt{m}}\right]$$

Find the velocity by taking the derivative with respect to time.

$$\dot{x} -> \frac{\sqrt{k}\, C[1] \, \text{Cos}\left[\frac{\sqrt{k}\, t}{\sqrt{m}}\right]}{\sqrt{m}} - \frac{\sqrt{k}\, C[2] \, \text{Sin}\left[\frac{\sqrt{k}\, t}{\sqrt{m}}\right]}{\sqrt{m}}$$

The Damped Simple Harmonic Oscillator

Generate the differential equation for a damped simple harmonic oscillator as done above.

$$\left(\frac{d}{dt}, \vec{0}\right)^2 (0, x, 0, 0) + \left(\frac{d}{dt}, \vec{0}\right) (0, b\, x, 0, 0) + \left(0, \frac{k}{m}x, 0, 0\right) =$$

$$= \left(0, \frac{d^2 x}{dt^2} + \frac{b\, d\, x}{dt} + \frac{k\, x}{m}, 0, 0\right) = 0$$

Solve the equation.

$$x -> C[1] \, e^{\frac{\left(-b\, m - \sqrt{-4\, k\, m + b^2\, m^2}\right) t}{2\, m}} + C[2] \, e^{\frac{\left(-b\, m + \sqrt{-4\, k\, m + b^2\, m^2}\right) t}{2\, m}}$$

The Wave Equation

Consider a wave traveling along the x direction. The equation which governs its motion is given by

$$\left(\frac{d}{v\, dt}, \frac{d}{dx}, 0, 0\right)^2 (0, 0, f[t\, v + x], 0) =$$

$$= \left(0, 0, \left(-\frac{d^2}{dx^2} + \frac{d^2}{dt^2\, v^2}\right) f[t\, v + x], \frac{2\, d^2\, f[t\, v + x]}{dt\, dx\, v}\right)$$

The third term is the one dimensional wave equation. The forth term is the instantaneous power transmitted by the wave.

Implications

Using the appropriate combinations of quaternion operators, the classical simple harmonic oscillator and wave equation were written out and solved. The functional definition of classical physics employed here is that the time operator is decoupled from any space operator. There is no reason why a similar combination of operators cannot be used when time and space operators are not decoupled. In fact, the four Maxwell equations appear to be one nonhomogeneous quaternion wave equation, and the structure of the simple harmonic oscillator appears in the Klein-Gordon equation.

3 Four Tests for a Conservative Force

There are four well-known, equivalent tests to determine if a force is conservative: the curl is zero, a potential function whose gradient is the force exists, all closed path integrals are zero, and the path integral between any two points is the same no matter what the path chosen. In this notebook, quaternion operators perform these tests on quaternion-valued forces.

1. The Curl Is Zero

To make the discussion concrete, define a force quaternion F.

$$F = (0, -k x, -k y, 0)$$

The curl is the commutator of the differential operator and the force. If this is zero, the force is conservative.

$$\text{odd}\left(\left(\frac{d}{dt}, \vec{\nabla}\right), \vec{F}\right) = 0$$

Let the differential operator quaternion act on the force, and test if the vector components equal zero.

$$\left(\frac{d}{dt}, \nabla\right) F = (2 k, 0, 0, 0)$$

2. There Exists a Potential Function for the Force

Operate on force quaternion using integration. Take the negative of the gradient of the first component. If the field quaternion is the same, the force is conservative.

$$F = \int F(dt, dx, dy, dz) =$$

$$= \int (k x\, dx + k y\, dy, -k x\, dt + k y\, dz, -k y\, dt - k x\, dz, 0) =$$

$$= \left(\frac{k x^2}{2} + \frac{k y^2}{2}, -k t x + k y z, -k t y - k x z, 0\right) =$$

$$\left(\frac{d}{dt}, \vec{\nabla}\right)\left(\frac{k x^2}{2} + \frac{k y^2}{2}, \vec{0}\right) = (0, -k x, -k y, 0)$$

This is the same force as we started with, so the scalar inside the integral is the scalar potential of this vector field. The vector terms inside the integral arise as constants of integration. They are zero if t=z=0. What role these vector terms in the potential quaternion may play, if any, is unknown to me.

3. The Line Integral of Any Closed Loop Is Zero

Use any parameterization in the line integral, making sure it comes back to go.

$$\text{path} = (0, r \operatorname{Cos}(t), r \operatorname{Sin}(t), 0)$$

$$\int_0^{2\pi} F\, dt = 0$$

4. The Line Integral Along Different Paths Is the Same

Choose any two parameterizations from A to B, and test that they are the same. These paths are from (0, r, 0, 0) to (0, -r, 2 r, 0).

$$\text{path1} = \left(0,\ r\,\text{Cos}(t),\ 2\,r\,\text{Sin}\left(\frac{t}{2}\right),\ 0\right)$$

$$\int_0^{2\pi} dt = -2\,k\,r^2$$

$$\text{path2} = (0,\ -t\,r + r,\ t\,r,\ 0)$$

$$\int_0^{2} F\,dt = -2\,k\,r^2$$

The same!

Implications

The four standard tests for a conservative force can be done with operator quaternions. One new avenue opened up is for doing path integrals. It would be interesting to attempt four dimensional path integrals to see where that might lead!