Quaternionic Analysis

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1 Introduction

The richness of the theory of functions over the complex field makes it natural to look for a similar theory for the only other non-trivial real associative division algebra, namely the quaternions. Such a theory exists and is quite far-reaching, yet it seems to be little known. It was not developed until nearly a century after Hamilton's discovery of quaternions. Hamilton himself [1] and his principal followers and expositors, Tait [2] and Joly [3], only developed the theory of functions of a quaternion variable as far as it could be taken by the general methods of the theory of functions of several real variables (the basic ideas of which appeared in their modern form for the first time in Hamilton's work on quaternions). They did not delimit a special class of regular functions among quaternion-valued functions of a quaternion variable, analogous to the regular functions of a complex variable.

This may have been because neither of the two fundamental definitions of a regular function of a complex variable has interesting consequences when adapted to quaternions; one is too restrictive, the other not restrictive enough. The functions of a quaternion variable which have quaternionic derivatives, in the obvious sense, are just the constant and linear functions (and not all of them); the functions which can be represented by quaternionic power series are just those which can be represented by power series in four real variables.

In 1935 R Fueter [4] proposed a definition of "regular" for quaternionic functions by means of an analogue of the Cauchy-Riemann equations. He showed that this definition led to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion [5]. In the next twelve years Fueter and his collaborators developed the theory of quaternionic analysis. A complete bibliography of this work is contained in ref. [6], and a simple account (in English) of the elementary parts of the theory has been given by Deavours [7].

The theory developed by Fueter and his school is incomplete in some ways, and many of their theorems are neither so general nor so rigorously proved as present-day standards of exposition in complex analysis would require. The purpose of this paper is to give a self-contained account of the main line of quaternionic analysis which remedies these deficiencies, as well as adding a certain number of new results. By using the exterior differential calculus we are able to give new and simple proofs of most of the main theorems and to clarify the relationship between quaternionic analysis and complex analysis.

Let $\mathbb{H}$ denote the algebra of quaternions, and let $\{1, i, j, k\}$ be an orthonormal basis, with the product on $\mathbb{H}$ given by the usual multiplication table (see section 2). The typical quaternion can be written as

$$q = t + ix + jy + kz \tag{1.1}$$

where t, x, y, z are real coordinates. Fueter defined a function $f : \mathbb{H} \to \mathbb{H}$ to be *regular* if it satisfied the equation

$$\frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z} = 0. \tag{1.2}$$

This is an analogue of the Cauchy-Riemann equations for a complex function $f : \mathbb{C} \to \mathbb{C}$, which can be written as

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0, \tag{1.3}$$

where the variable is $z = x + iy$.

Fueter showed that any quaternionic function which is regular and also continuously differentiable must satisfy an analogue of Cauchy's theorem which can be written as

$$\int_C Dq\, f = 0, \tag{1.4}$$

where C is any smooth closed 3-manifold in $\mathbb{H}$ and Dq is a certain natural quaternion-valued 3-form. Dq is defined in section 2; it can be thought of as the quaternion representing an element δC of the 3-manifold, its magnitude being equal to the volume of δC and its direction being normal to δC. Fueter also obtained an analogue of Cauchy's integral formula for such functions, namely

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial D} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\, f(q), \tag{1.5}$$

where D is a domain in $\mathbb{H}$ in which f is regular and q_0 is a point inside D.

The complex Cauchy-Riemann equation (1.3) is equivalent to the statement that f has a complex derivative, i.e. that there exists a complex number $f'(z)$ such that

$$df = f'(z)\, dz. \tag{1.6}$$

Fueter gave no corresponding characterisation of a regular function of a quaternion variable, leaving the theorems (1.4) and (1.5) and the analogy with the Cauchy-Riemann equations as sufficient justification of the definition (1.2). In this paper we will show that a regular function can be defined as one which has a certain kind of quaternionic derivative; specifically, (1.2) is equivalent to the existence of a quaternion $f'(q)$ such that

$$d(dq \wedge dq\, f) = Dq\, f'(q). \tag{1.7}$$

(the 2-form $dq \wedge dq$ is described in section 2).

Cauchy's theorem (1.4) and the integral formula (1.5) can be simply proved by showing that

$$d(Dqf) = 0 \tag{1.8}$$

and

$$d\left[\frac{(q-q_0)^{-1}}{|q-q_0|^2}\right] Dq\, f(q) = \Delta\left(\frac{1}{|q-q_0|^2}\right) f(q) v \tag{1.9}$$

where Δ is the Laplacian on $\mathbb{R}^4$ and $v = dt \wedge dx \wedge dy \wedge dz$ is the standard volume 4-form. Since $|q - q_0|^{-2}$ is the Green's function for the Laplacian in $\mathbb{R}^4$, (1.5) follows from (1.9). This is essentially how Fueter proved these theorems. Since both proofs use Stokes's theorem, they need the condition that the partial derivatives of the function should be continuous. Schuler [8] showed that this condition could be dropped by adapting Goursat's proof of Cauchy's theorem, but he did not draw the full consequences of this argument. In fact Cauchy's theorem (1.4) can be proved for any rectifiable contour C and any function f which is differentiable at each point inside C and whose partial derivatives satisfy (1.2). The integral formula (1.5) has a similarly wide range of validity. From this it follows, as in complex analysis, that if f is regular in an open set U then it has a power series expansion about each point of U. Thus pointwise differentiability, together with the four real conditions (1.2) on the sixteen partial derivatives of f, is sufficient to ensure analyticity.

The homogeneous components in the power series representing a regular function are themselves regular; thus it is important to study regular homogeneous polynomials, the basic regular functions from which all regular functions are constructed. The corresponding functions of a complex variable are just the powers of the variable, but the situation with quaternions is more complicated. The set of homogeneous regular functions of degree n forms a quaternionic vector space of dimension $\frac{1}{2}(n+1)(n+2)$; this is true for any integer n if for negative n it is understood that the functions are defined and regular everywhere except at 0. The functions with negative degree of homogeneity correspond to negative powers of a complex variable, and occur in the quaternionic Laurent series which exists for any regular function which is regular in an open set except at one point. Fueter found two natural bases for the set of homogeneous functions, which play dual roles in the calculus of residues. (He actually only proved that these bases formed spanning sets). In this paper we will study homogeneous regular functions by means of harmonic analysis on the unit sphere in $\mathbb{H}$, which forms a group isomorphic to SU(2); this bears the same relation to quaternionic analysis as the theory of Fourier series does to complex analysis.

Many of the algebraic and geometric properties of complex analytic functions are not present in quaternionic analysis. Because quaternions do not commute, regular functions of a quaternion variable cannot be multiplied or composed to give further regular functions. Because the quaternions are four-dimensional, there is no counterpart to the geometrical description of complex analytic functions as conformal mappings. The zeros of a quaternionic regular function are not necessarily isolated, and its range is not necessarily open; neither of these sets need even be a submanifold of $\mathbb{H}$. There is a corresponding complexity in the structure of the singularities of a quaternionic regular function; this was described by Fueter [9], but without giving precise statements or proofs. This topic is not investigated here.

The organisation of this paper is as follows.

In section 2 the basic algebraic facts about quaternions are surveyed and notation is established; some special algebraic concepts are introduced, and quaternionic differential forms are described.

Section 3 is concerned with the definition of a regular function. The remarks in the second paragraph of this introduction, about possible analogues of complex definitions of analyticity, are amplified (this material seems to be widely known, but is not easily accessible in the literature), and the definition (1.7) of regular functions by means of the quaternionic derivative is shown to be equivalent to Fueter's definition (1.2) by means of a Cauchy-Riemann-type equation.

Section 4 is devoted to the analogues of the Cauchy-Goursat theorem and Cauchy's integral formula.

Section 5 contains analogues of Liouvilles's theorem, the maximum-modulus principle, and Morera's theorem. After the work of section 4, only the last of these requires proof.

In section 6 we show how regular functions can be constructed from functions of more familiar type, namely harmonic functions of four real variables and analytic functions of a complex variable.

Section 7 is concerned with the effect on regular functions of conformal transformation of the variable; these results are needed in the last two sections.

Section 8 is an investigation of homogeneous regular functions by means of harmonic analysis on S^3.

In section 9 we examine the power series representing a regular function and prove analogues of Laurent's Theorem and the residue theorem.

2 Preliminaries

2.1 The Algebra of Quaternions[1]

The quaternions $\mathbb{H}$ form a four-dimensional algebra over the real field $\mathbb{R}$, with an identity element denoted by 1. We regard $\mathbb{R}$ as being embedded in $\mathbb{H}$ by identifying $t \in \mathbb{R}$ with $t1 \in \mathbb{H}$. Then we can write $\mathbb{H} = \mathbb{R} \oplus P$, where P is an oriented three-dimensional Euclidean vector space, and the product of two quaternions is defined by

$$(a_0, \mathbf{a})(b_0, \mathbf{b}) = (a_0 b_0 - \mathbf{a}.\mathbf{b}, a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \tag{2.1}$$

where $a_0, b_0 \in \mathbb{R}$, $\mathbf{a}, \mathbf{b} \in P$, $\mathbf{a}.\mathbf{b}$ denotes the inner product of $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ denotes the vector product determined by the orientation on P. (Conversely, the subspace P, its inner product and its orientation can be defined in terms of the multiplication on $\mathbb{H}$.)

Thus we can choose a basis $\{1, i, j, k\}$ for $\mathbb{H}$ so that the multiplication is given by

$$\left.\begin{array}{c} i^2 = j^2 = k^2 = -1, \\ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \end{array}\right\} \tag{2.2}$$

The typical quaternion will be denoted by

$$q = t + xi + yj + zk \qquad (t, x, y, z \in \mathbb{R}) \tag{2.3}$$

In performing calculations it is sometimes useful to denote the basic quaternions i, j, k by e_i $(i = 1, 2, 3)$ and the coordinates x, y, z by x_i $(i = 1, 2, 3)$, and to use the summation convention for repeated indices. In this notation eq. (2.3) becomes

$$q = t + x_i e_i \tag{2.4}$$

and the multiplication rules (2.2) become

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \tag{2.5}$$

[1]Proofs of the assertions in this section can be found in [10, chap. 10].

where ϵ_{ijk} is the usual alternating symbol.

The centre of the algebra $\mathbb{H}$ is the real subfield $\mathbb{R}$. If q is any quaternion, the vector subspace spanned by 1 and q is a subfield of $\mathbb{H}$, which is isomorphic to the complex numbers $\mathbb{C}$ if 1 and q are linearly independent. It is sometimes convenient to distinguish a particular embedding of $\mathbb{C}$ in $\mathbb{H}$; whenever we want to do this, we will take $\mathbb{C}$ to be the subfield spanned by 1 and i. Then any quaternion can be written as

$$q = v + jw \tag{2.6}$$

with $v, w \in \mathbb{C}$; if q is given by (2.3), then

$$v = t + ix \quad \text{and} \quad w = y - iz \tag{2.7}$$

From an equation of the form $v_1 + jw_1 = v_2 + jw_2$ with $v_1, v_2, w_1, w_2 \in \mathbb{C}$, we can deduce that $v_1 = v_2$ and $w_1 = w_2$. When quaternions are written in this form, the basic law of multiplication is

$$vj = j\bar{v}. \tag{2.8}$$

The *conjugate* of the quaternion q is

$$\bar{q} = t - ix - jy - kz. \tag{2.9}$$

Conjugation is an involutive anti-automorphism of $\mathbb{H}$, i.e. it is $\mathbb{R}$-linear, $\bar{\bar{q}} = q$, and

$$\overline{q_1 q_2} = \bar{q}_2\, \bar{q}_1. \tag{2.10}$$

For every embedding of C in $\mathbb{H}$, quaternion conjugation coincides with complex conjugation. q commutes with $\bar{q}$, and their product is

$$q\bar{q} = t^2 + x^2 + y^2 + z^2. \tag{2.11}$$

The *modulus* of q is the non-negative real number

$$|q| = \sqrt{q\bar{q}} \tag{2.12}$$

It follows from (2.11) that every non-zero quaternion has a multiplicative inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}. \tag{2.13}$$

The *real part* of q is

$$\mathrm{Re}\, q = t = \tfrac{1}{2}(q + \bar{q}) \in \mathbb{R} \tag{2.14}$$

and its *pure quaternion part* (or *vector part*) is

$$\mathrm{Pu}\, q = xi + yj + zk = \tfrac{1}{2}\,(q - \bar{q}) \in P. \tag{2.15}$$

It follows from (2.1) that

$$\mathrm{Re}(q_1 q_2) = \mathrm{Re}(q_2 q_1) \tag{2.16}$$

and from (2.9) that

$$\mathrm{Re}\, \bar{q} = \mathrm{Re}\, q. \tag{2.17}$$

q is a *unit quaternion* if $|q| = 1$. The set of unit quaternions, constituting the sphere S^3, forms a multiplicative Lie group isomorphic to SU(2); we will denote it by S. The *versor* of a quaternion q is the unit quaternion

$$\mathrm{Un}\, q = \frac{q}{|q|} \tag{2.18}$$

Any quaternion has a polar decomposition $q = ru$ where $r = |q| \in \mathbb{R}$ and $u = \mathrm{Un}\, q \in S$.

The classical notation for these functions of q is $\mathrm{S}\, q = \mathrm{Re}\, q$, $\mathrm{V}\, q = \mathrm{Pu}\, q$, $\mathrm{K}\, q = \bar{q}$, $\mathrm{T}\, q = |q|$, $\mathrm{U}\, q = \mathrm{Un}\, q$.

The positive-definite quadratic form (2.11) gives rise to an inner product

$$\begin{aligned} \langle q_1, q_2 \rangle &= \mathrm{Re}(q_1 \bar{q}_2) \\ &= \mathrm{Re}(\bar{q}_1 q_2) \quad \text{by (2.16) and (2.17)} \end{aligned} \tag{2.19}$$

Note that

$$\langle a q_1, q_2 \rangle = \langle q_1, \bar{a} q_2 \rangle \tag{2.20}$$

and

$$\langle q_1 a, q_2 \rangle = \langle q_1, q_2 \bar{a} \rangle, \tag{2.21}$$

i.e. the adjoint of left (right) multiplication by a is left (right) multiplication by $\bar{a}$, for any $a \in \mathbb{H}$. If u_1 and u_2 are unit quaternions, the map $q \mapsto u_1 q u_2$ is orthogonal with respect to this inner product and has determinant 1; conversely, any rotation of $\mathbb{H}$ is of the form $q \mapsto u_1 q u_2$ for some $u_1, u_2 \in S$. This is the well-known double cover $0 \to \mathbb{Z}_2 \to \mathrm{SU(2)} \times \mathrm{SU(2)} \to \mathrm{SO(4)} \to 0$.

This inner product induces an $\mathbb{R}$-linear map $\Gamma : \mathbb{H}^* \to \mathbb{H}$, where $\mathbb{H}^* = \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{R})$ is the dual of $\mathbb{H}$, given by

$$\langle \Gamma(\alpha), q \rangle = \alpha(q) \tag{2.22}$$

for $a \in \mathbb{H}^*$, $q \in \mathbb{H}$. Since $\{1, i, j, k\}$ is an orthonormal basis for $\mathbb{H}$, we have

$$\Gamma(\alpha) = \alpha(1) + i\, \alpha(i) + j\, \alpha(j) + k\, \alpha(k). \tag{2.23}$$

The set of $\mathbb{R}$-linear maps of $\mathbb{H}$ into Itself forms a two-sided vector space over $\mathbb{H}$ of dimension 4, which we will denote by F_1. It is spanned (over $\mathbb{H}$) by $\mathbb{H}^*$, so the map Γ can be extended by linearity to a right $\mathbb{H}$-linear map $\Gamma_r : F_1 \to \mathbb{H}$ and a left-linear map $\Gamma_\ell : F_1 \to \mathbb{H}$. They are given by

$$\Gamma_r(\alpha) = \alpha(1) + i\, \alpha(i) + j\, \alpha(j) + k\, \alpha(k) \tag{2.24}$$

and

$$\Gamma_\ell(\alpha) = \alpha(1) + \alpha(i)\, i + \alpha(j)\, j + \alpha(k)\, k \tag{2.25}$$

for any $\alpha \in F_1$.

The inner product (2.19) and the maps $\Gamma_r, \Gamma_\ell : F_1 = \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{R}) \to \mathbb{H}$ have alternative characterisations in terms of quaternion multiplication and conjugation; in fact they are obtained by slight modifications of standard procedures from the tensor of type (1,2) which defines the multiplication on $\mathbb{H}$. Let $L_q : \mathbb{H} \to \mathbb{H}$ and $R_q : \mathbb{H} \to \mathbb{H}$ be the operations of multiplication by q on the left and on the right, respectively, and let $K : \mathbb{H} \to \mathbb{H}$ be the operation of conjugation; then

Theorem 1

(i) $\langle q, p \rangle = -\frac{1}{2} \operatorname{Tr}(L_q R_p K)$

(ii) $\langle \Gamma_r(\alpha), q \rangle = \operatorname{Tr}(R_{\bar{q}} \alpha K)$

(iii) $\langle \Gamma_\ell(\alpha), q \rangle = \operatorname{Tr}(L_{\bar{q}} \alpha K)$ *for all* $p, q \in \mathbb{H}$ *and* $\alpha \in F_1$.

Proof The trace of an $\mathbb{R}$-linear map $\alpha : \mathbb{H} \to \mathbb{H}$ is

$$\operatorname{Tr} \alpha = \operatorname{Re}[\alpha(1)] - \sum_i \operatorname{Re}[e_i \alpha(e_i)] \tag{2.26}$$

Hence (i)

$$\operatorname{Tr}(L_q R_p K) = \operatorname{Re}\left[qp + \sum_i e_i q e_i p\right]$$

But

$$q + \sum_i e_i q e_i = -2\bar{q} \tag{2.27}$$

as can easily be verified, so

$$\operatorname{Tr}(L_q R_p K) = -2 \operatorname{Re}(\bar{q} p) = -2\langle q, p \rangle. \qquad \square$$

(ii)

$$\begin{aligned} \operatorname{Tr}(R_{\bar{q}} \alpha K) &= \operatorname{Re}\left[\alpha(1)\bar{q} + \sum_i e_i \alpha(e_i) \bar{q}\right] \\ &= \operatorname{Re}[\Gamma_r(\alpha)\bar{q}] \qquad \text{by (2.24)} \\ &= \langle \Gamma_r(\alpha), q \rangle \end{aligned} \qquad \square$$

(iii)

$$\begin{aligned} \operatorname{Tr}(L_{\bar{q}} \alpha K) &= \operatorname{Re}\left[\bar{q}\alpha(1) + \sum_i e_i \bar{q} \alpha(e_i)\right] \\ &= \operatorname{Re}\left[\bar{q}\alpha(1) + \sum_i \bar{q} \alpha(e_i) e_i\right] \qquad \text{by (2.16)} \\ &= \operatorname{Re}[\bar{q}\Gamma_\ell(\alpha)] \qquad \text{by (2.25)} \\ &= \langle \Gamma_\ell(\alpha), q \rangle \qquad \square \end{aligned}$$

We will also need two other maps $\bar{\Gamma}_r, \bar{\Gamma}_\ell : F_1 \to \mathbb{H}$, defined as follows:

$$\langle \bar{\Gamma}_r(\alpha), q \rangle = \operatorname{Tr}(R_{\bar{q}} \alpha); \qquad \bar{\Gamma}_r(\alpha) = \alpha(1) - \sum_i e_i \alpha(e_i) \tag{2.28}$$

$$\langle \bar{\Gamma}_\ell(\alpha), q \rangle = \operatorname{Tr}(L_q \alpha); \qquad \bar{\Gamma}_\ell(\alpha) = \alpha(1) - \sum_i \alpha(e_i) e_i \tag{2.29}$$

2.2 Quaternionic Differential Forms

When it is necessary to avoid confusion with other senses of differentiability which we will consider, we will say that a function $f : \mathbb{H} \to \mathbb{H}$ is *real-differentiable* if it is differentiable in the usual sense. Its differential at a point $q \in \mathbb{H}$ is then an $\mathbb{R}$-linear map $df_q : \mathbb{H} \to \mathbb{H}$. By identifying the tangent space at each point of $\mathbb{H}$ with $\mathbb{H}$ itself, we can regard the differential as a quaternion-valued 1-form

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz. \tag{2.30}$$

Conversely, any quaternion-valued 1-form $\theta = a_0\,dt + a_i\,dx_i$ ($a_0, a_i \in \mathbb{H}$) can be regarded as the $\mathbb{R}$-linear map $\theta : \mathbb{H} \to \mathbb{H}$ given by

$$\theta(t + x_i e_i) = a_0 t + a_i x_i \tag{2.31}$$

Similarly, a quaternion-valued r-form can be regarded as a mapping from $\mathbb{H}$ to the space of alternating $\mathbb{R}$-multilinear maps from $\mathbb{H} \times ... \times \mathbb{H}$ (r times) to $\mathbb{H}$. We define the exterior product of such forms in the usual way: if θ is an r-form and ϕ is an s-form,

$$\theta \wedge \phi(h_1, ..., h_{r+s}) = \tfrac{1}{r!s!} \sum_\rho \epsilon(\rho)\theta(h_{\rho(1)}, ..., h_{\rho(r)})\phi(h_{\rho(r+1)}, ..., h_{\rho(r+s)}), \tag{2.32}$$

where the sum is over all permutations ρ of $r + s$ objects, and $\epsilon(\rho)$ is the sign of ρ. Then the set of all r-forms is a two-sided quaternionic vector space, and we have

$$\left.\begin{aligned} a(\theta \wedge \phi) &= (a\theta) \wedge \phi, \\ (\theta \wedge \phi)a &= \theta \wedge (\phi a), \\ (\theta a) \wedge \phi &= \theta \wedge (a\phi) \end{aligned}\right\} \tag{2.33}$$

for all quaternions a, r-forms θ and s-forms ϕ. The space of quaternionic r-forms has a basis of real r-forms, consisting of exterior products of the real 1-forms dt, dx, dy, dz; for such forms left and right multiplication by quaternions coincide. Note that because the exterior product is defined in terms of quaternion multiplication, which is not commutative, it is not in general true that $\theta \wedge \phi = -\phi \wedge \theta$ for quaternionic 1-forms θ and ϕ.

The exterior derivative of a quaternionic differential form is defined by the usual recursive formulae, and Stokes's theorem holds in the usual form for quaternionic integrals.

The following special differential forms will be much used in the rest of the paper. The differential of the identity function is

$$dq = dt + i\,dx + j\,dy + k\,dz; \tag{2.34}$$

regarded as $\mathbb{R}$-linear transformation of $\mathbb{H}$, dq is the identity mapping. Its exterior product with itself is

$$dq \wedge dq = \tfrac{1}{2}\epsilon_{ijk}\, e_i\, dx_j \wedge dx_k = i\, dy \wedge dz + j\, dz \wedge dx + k\, dx \wedge dy \tag{2.35}$$

which, as antisymmetric function on $\mathbb{H} \times \mathbb{H}$, gives the commutator of its arguments. For the (essentially unique) constant real 4-form we use the abbreviation

$$v = dt \wedge dx \wedge dy \wedge dz, \tag{2.36}$$

so that $v(1, i, j, k) = 1$. Finally, the 3-form Dq is defined as an alternating $\mathbb{R}$-trilinear function by

$$\langle h_1, Dq(h_2, h_3, h_4) \rangle = v(h_1, h_2, h_3, h_4) \tag{2.37}$$

for all $h_1, ..., h_4 \in \mathbb{H}$. Thus $Dq(i, j, k) = 1$ and $Dq(1, e_i, e_j) = -\epsilon_{ijk} e_k$. The coordinate expression for Dq is

$$\begin{aligned} Dq &= dx \wedge dy \wedge dz - \tfrac{1}{2}\epsilon_{ijk}\, e_i\, dt \wedge dx_j \wedge dx_k \\ &= dx \wedge dy \wedge dz - i\, dt \wedge dy \wedge dz - j\, dt \wedge dz \wedge dx - k\, dt \wedge dx \wedge dy. \end{aligned} \tag{2.38}$$

Geometrically, $Dq(a, b, c)$ is a quaternion which is perpendicular to a, b and c and has magnitude equal to the volume of the 3-dimensional parallelepiped whose edges are a, b and c. It also has the following algebraic expression:

Theorem 2 $Dq(a, b, c) = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$

Proof For any unit quaternion u, the map $q \mapsto uq$ is an orthogonal transformation of $\mathbb{H}$ with determinant 1; hence

$$Dq(ua, ub, uc) = u\, Dq(a, b, c).$$

Taking $u = |a|^{-1}$, and using the $\mathbb{R}$-trilinearity of Dq, we obtain

$$Dq(a, b, c) = |a|^2 a\, Dq(1, a^{-1}b, a^{-1}c). \tag{2.39}$$

Now since $Dq(1, e_i, e_j) = -\epsilon_{ijk} e_k = \frac{1}{2}(e_j e_i - e_i e_j)$, we have by linearity

$$Dq(1, h_1, h_2) = \tfrac{1}{2}(h_2 h_1 - h_1 h_2) \tag{2.40}$$

for all $h_1, h_2 \in \mathbb{H}$. Hence

$$\begin{aligned} Dq(a, b, c) &= \tfrac{1}{2}|a|^2 a(a^{-1}ca^{-1}b - a^{-1}ba^{-1}c) \\ &= \tfrac{1}{2}(c\bar{a}b - b\bar{a}c) \quad \text{by (2.13).} \qquad \square \end{aligned}$$

Two useful formulae were obtained in the course of this proof. The argument leading to (2.39) can be generalised, using the fact that the map $q \mapsto uqv$ is a rotation for any pair of unit quaternions u, v, to

$$Dq(ah_1b, ah_2b, ah_3b) = |a|^2|b|^2 a\, Dq(h_1, h_2, h_3) b; \tag{2.41}$$

and the formula (2.40) can be written as

$$1 \rfloor Dq = -\tfrac{1}{2}\, dq \wedge dq, \tag{2.42}$$

where $\rfloor$ denotes the usual inner product between differential forms and vector fields and 1 denotes the constant vector field whose value is 1.

Since the differential of a quaternion-valued function on $\mathbb{H}$ is an element of F_1, the map Γ_r can be applied to it. The result is

$$\Gamma_r(df) = \frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z}. \tag{2.43}$$

We introduce the following notation for the differential operator occurring in (2.40), and for other related differential operators:

$$\left.\begin{aligned}
\bar{\partial}_\ell f &= \tfrac{1}{2}\Gamma_r(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} + e_i\frac{\partial f}{\partial x_i}\right),\\
\partial_\ell f &= \tfrac{1}{2}\bar{\Gamma}_r(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} - e_i\frac{\partial f}{\partial x_i}\right),\\
\bar{\partial}_r f &= \tfrac{1}{2}\Gamma_\ell(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}e_i\right),\\
\partial_r f &= \tfrac{1}{2}\bar{\Gamma}_\ell(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x_i}e_i\right),\\
\Delta f &= \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}\right\} \tag{2.44}$$

Note that $\partial_\ell, \bar{\partial}_\ell, \partial_r$ and $\bar{\partial}_r$ all commute, and that

$$\Delta = 4\partial_r\bar{\partial}_r = 4\partial_\ell\bar{\partial}_\ell \tag{2.45}$$

3 Regular Functions

We start by showing that the concept of an analytic function of a quaternion variable as one which is constructed from the variable by quaternion addition and multiplication (possibly involving infinite series) is the same as the concept of a real-analytic function in the four real variables t, x, y, z.

Definition 1 *A* **quaternionic monomial** *is a function $f : \mathbb{H} \to \mathbb{H}$ of the form*

$$f(q) = a_0 q a_1 q ... q_{r-1} q a_r \tag{3.1}$$

for some non-negative integer r (the **degree** *of the monomial) and constant quaternions $a_0, ..., a_r$.*

A **quaternionic polynomial** *is a finite sum of quaternionic monomials.*

A **homogeneous polynomial function** *of degree r on $\mathbb{H}$ is a function $f : \mathbb{H} \to \mathbb{H}$ of the form*

$$f(q) = F(q, ..., q)$$

where $F : \mathbb{H} \times ... \times \mathbb{H}$ (r times) $\to \mathbb{H}$ is $\mathbb{R}$-multilinear.

A **polynomial function** *on $\mathbb{H}$ is a finite sum of homogeneous polynomial functions of varying degrees.*

Theorem 3 *Every polynomial function on $\mathbb{H}$ is a quaternionic polynomial.*

Proof Any polynomial function f can be written as

$$f(q) = f_0(t, x, y, z) + \sum_i f_i(t, x, y, z)e_i$$

where f_0 and f_i are four real-valued polynomials in the four real variables t, x, y, z. But

$$\left.\begin{aligned} t &= \tfrac{1}{4}(q - iqi - jqj - kqk), \\ x &= \tfrac{1}{4i}(q - iqi + jqj + kqk), \\ y &= \tfrac{1}{4j}(q + iqi - jqj + kqk), \\ z &= \tfrac{1}{4k}(q + iqi + jqj - kqk). \end{aligned}\right\} \tag{3.2}$$

Putting these expressions for t, x, y, z into the polynomials f_0, f_i, we obtain $f(q)$ as a sum of expressions in q of the form (3.1), so f is a quaternionic polynomial. □

It is clear that conversely, every quaternionic polynomial is a polynomial function, so we have

Corollary The class of functions which are defined in a neighbourhood of the origin in $\mathbb{H}$ and can be represented there by a quaternionic power series, i.e. a series of quaternionic monomials, is precisely the same as the class of functions which are real-analytic in a neighbourhood of the origin.

We now turn to an alternative attempt to parallel complex analysis, in which we concentrate on the existence of a quaternionic derivative defined as the limit of a difference quotient. We will show that only quaternionic polynomials of degree 1 (and not all of them) possess such a derivative.

Definition 2 *A function $f : \mathbb{H} \to \mathbb{H}$ is* **quaternion-differentiable on the left** *at q if the limit*

$$\frac{df}{dq} = \lim_{h\to 0} \left[h^{-1}\{f(q + h) - f(q)\}\right]$$

exists.

Theorem 4 *Suppose the function f is defined and quaternion-differentiable on the left throughout a connected open set U. Then on U, f has the form*

$$f(q) = a + qb$$

for some $a, b \in \mathbb{H}$.

Proof From the definition it follows that if f is quaternion-differentiable on the left at q, it is real-differentiable at q and its differential is the linear map of multiplication on the right by $\frac{df}{dq}$:

$$df_q(h) = h\,\frac{df}{dq},$$

i.e.

$$df_q = dq\,\frac{df}{dq}.$$

Equating coefficients of dt, dx, dy and dz gives

$$\frac{df}{dq} = \frac{\partial f}{\partial t} = -i\,\frac{\partial f}{\partial x} = -j\,\frac{\partial f}{\partial y} = -k\,\frac{\partial f}{\partial z}. \tag{3.3}$$

Put $q = v + jw$, where $v = t + ix$ and $w = y - iz$, and let $f(q) = g(v, w) + jh(v, w)$, where g and h are complex-valued functions of the two complex variables v and w; then (3.3) can be separated into the two sets of complex equations

$$\frac{\partial g}{\partial t} = -i\,\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} = i\,\frac{\partial h}{\partial z},$$

$$\frac{\partial h}{\partial t} = i\,\frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y} = i\,\frac{\partial g}{\partial z}.$$

In terms of complex derivatives, these can be written as

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial w} = 0, \tag{3.4}$$

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial w}, \tag{3.5}$$

and

$$\frac{\partial h}{\partial \bar{v}} = -\frac{\partial g}{\partial \bar{w}}. \tag{3.6}$$

Eq. (3.4) shows that g is a complex-analytic function of v and $\bar{w}$, and h is a complex-analytic function of $\bar{v}$ and w. Hence by Hartogs's theorem [11, p. 133] g and h have continuous partial derivatives of all orders and so from (3.5)

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial h}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial h}{\partial v}\right) = 0.$$

Suppose for the moment that U is convex. Then we can deduce that g is linear in $\bar{w}$, h is linear in w and h is linear in $\bar{v}$. Thus

$$g(v, w) = \alpha + \beta v + \gamma \bar{w} + \delta v \bar{w},$$
$$h(v, w) = \epsilon + \zeta \bar{v} + \eta w + \theta \bar{v} w,$$

where the Greek letters represent complex constants. Now (3.5) and (3.6) give the following relations among these constants:

$$\beta = \eta, \quad \zeta = -\gamma, \quad \delta = \theta = 0.$$

Thus

$$\begin{aligned} f = g + jh &= \alpha + j\epsilon + (v + jw)(\beta - j\gamma) \\ &= a + qb, \end{aligned}$$

where $a = \alpha + j\epsilon$ and $b = \beta - j\gamma$; so f is of the stated form if U is convex. The general connected open set can be covered by convex sets, any two of which can be connected by a chain of convex sets which overlap in pairs; comparing the forms of the function

f on the overlaps, we see that $f(q) = a + qb$ with the same constants a, b throughout U. $\square$

Even if f is quaternion-differentiable, it will not in general satisfy Cauchy's theorem in the form

$$\int dqf = 0 \tag{3.7}$$

where the integral is round a closed curve; in fact the only functions satisfying this equation for all closed curves are the constant functions. We will prove this for the infinitesimal form of (3.7), namely

$$d(dqf) = 0. \tag{3.8}$$

Theorem 5 *If the function $f : U \to \mathbb{H}$ is real-differentiable in the connected open set U and satisfies $d(dqf) = 0$ in U, then f is constant on U.*

Proof

$$\begin{aligned} d(dqf) = dq \wedge df &= (dt + e_i dx_i) \wedge \left(\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_j}dx\right) \\ &= \left(\frac{\partial f}{\partial x_i} - e_i\frac{\partial f}{\partial t}\right) dt \wedge dx_i + e_i\frac{\partial f}{\partial x_j}dx_i \wedge dx_j \end{aligned}$$

Thus

$$d(dqf) = 0 \Rightarrow \frac{\partial f}{\partial x_i} = e_i\frac{\partial f}{\partial t} \tag{3.9}$$

and

$$e_i\frac{\partial f}{\partial x_j} = e_j\frac{\partial f}{\partial x_i} \tag{3.10}$$

From (3.9) we have, for example,

$$\frac{\partial f}{\partial x}i = \frac{\partial f}{\partial y}j$$

while from (3.10),

$$\frac{\partial f}{\partial x}j = \frac{\partial f}{\partial y}i.$$

Hence

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}ji = \frac{\partial f}{\partial y}ij = -\frac{\partial f}{\partial x}.$$

So $\frac{\partial f}{\partial x} = 0$ and therefore $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ in U. It follows that f is constant in U. $\square$

We will now give a definition of "regular" for a quaternionic function which is satisfied by a large class of functions and opens the door to a development similar to the theory of regular functions of a complex variable.

Definition 3 *A function $f : \mathbb{H} \to \mathbb{H}$ is* **left-regular** *at $q \in \mathbb{H}$ if it is real-differentiable at q and there exists a quaternion $f'_\ell(q)$ such that*

$$d(dq \wedge dq\, f) = -2Dq\, f'_\ell(q). \tag{3.11}$$

It is **right-regular** *if there exists a quaternion $f'_r(q)$ such that*

$$d(f\, dq \wedge dq) = -2f'_r(q)\, Dq.$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For definiteness, we will only consider left-regular functions, which we will call simply **regular**. *We will write $f'(q) = f'_\ell(q)$ and call it the* **derivative** *of f at q.*

Theorem 6 (the Cauchy-Riemann-Fueter equations)
A real-differentiable function f is regular at q if and only if

$$\Gamma_r(df_q) = 0. \tag{3.12}$$

Proof Suppose f is regular at q. Then from (3.11),

$$dq \wedge dq \wedge df_q = -2Dqf'(q)$$

Evaluating these trilinear functions with i, j, k as arguments gives

$$(ij - ji)\, df_q(k) + (jk - kj)\, df_q(i) + (ki - ik)\, df_q(j) = -2f'(q)$$

while using 1, i, j as arguments gives

$$(ij - ji)\, df_q(1) = 2kf'(q).$$

Hence

$$f'(q) = df_q(1) = -\{i\, df_q(i) + j\, df_q(j) + k\, df_q(k)\} \tag{3.13}$$

Comparing with (2.24), we see that $\Gamma_r(df_q) = 0$.

Conversely, if $\Gamma_r(df_q) = 0$ we can define $f'(q) = df_q(1)$ and then evaluating as above shows that $dq \wedge dq \wedge df_q = 2Dqf'(q)$, so that f is regular at q. □

Note that (3.12) and (3.13) can be written as

$$2\bar{\partial}_\ell f = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0 \tag{3.14}$$

and

$$f'(q) = \frac{\partial f}{\partial t}. \tag{3.15}$$

Hence $f' = \partial_\ell f$. If we write $q = v + jw$, $f(q) = g(v, w) + jh(v, w)$ as in theorem 4, (3.14) becomes the pair of complex equations

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}}, \qquad \frac{\partial g}{\partial w} = -\frac{\partial h}{\partial v} \tag{3.16}$$

which show some similarity to the Cauchy-Riemann equations for a function of a complex variable.

From (3.14) and (2.45) it follows that if f is regular and twice differentiable, then

$$\Delta f = 0,$$

i.e. f is harmonic. We will see in the next section that a regular function is necessarily infinitely differentiable, so all regular functions are harmonic.

The derivative of a regular function can be characterised as the limit of a difference quotient which is analogous to that used to define the derivative of a complex-analytic function.

Definition 4 *An* **oriented** k**-parallelepiped** *in* $\mathbb{H}$ *is a map* $C : I^k \to \mathbb{H}$*, where* $I^k \subset \mathbb{R}^k$ *is the closed unit* k*-cube, of the form*

$$C(t_1, ..., t_k) = q_0 + t_1 h_1 + ... + t_k h_k.$$

$q_0 \in \mathbb{H}$ *is called the* **original vertex** *of the parallelepiped, and* $h_1, ..., h_k \in \mathbb{H}$ *are called its* **edge-vectors***. A* k*-parallelepiped is* **non-degenerate** *if its edge-vectors are linearly independent (over* $\mathbb{R}$*). A non-degenerate 4-parallelepiped is* **positively oriented** *if* $v(h_1, ..., h_4) > 0$*,* **negatively oriented** *if* $v(h_1, ..., h_4) < 0$*.*

Theorem 7 *Suppose that* f *is regular at* q_0 *and continuously differentiable in a neighbourhood of* q_0*. Then given* $\epsilon > 0$*, there exists* $\delta > 0$ *such that if* C *is a non-degenerate oriented 3-parallelepiped with* $q_0 \in C(I^3)$ *and* $q \in C(I^3) \Rightarrow |q - q_0| < \delta$*, then*

$$\left| \left(\int_C Dq \right)^{-1} \left(\int_{\partial C} dq \wedge dqf \right) + 2f'(q_0) \right| < \epsilon.$$

Proof First note that $||h_1 \wedge h_2 \wedge h_3|| = |Dq(h_1, h_2, h_3)|$ is a norm on the real vector space $\mathbb{H} \wedge \mathbb{H} \wedge \mathbb{H}$. Now since df_q is a continuous function of q at q_0, so is $dq \wedge dq \wedge df_q$; hence we can choose δ so that

$$\begin{aligned} |q - q_0| < \delta \Rightarrow &|dq \wedge dq \wedge df_q(h_1, h_2, h_3) - dq \wedge dq \wedge df_{q_0}(h_1, h_2, h_3)| \\ &< \epsilon |Dq(h_1, h_2, h_3)|, \quad \forall h_1, h_2, h_3 \in \mathbb{H}. \end{aligned}$$

Let C be as in the statement of the theorem, with edge-vectors h_1, h_2, h_3. Then by Stokes's theorem,

$$\int_{\partial C} dq \wedge dqf = \int_C d(dq \wedge dqf) = \int\int\int_0^1 dq \wedge dq \wedge df_{C(t)}(h_1, h_2, h_3)\, dt_1 dt_2 dt_3$$

Since f is regular at q_0,

$$\begin{aligned} 2\int_C Dqf'(q_0) &= 2\int\int\int_0^1 Dq(h_1, h_2, h_3)\, f'(q_0)\, dt_1 dt_2 dt_3 \\ &= -\int\int\int_0^1 dq \wedge dq \wedge df_{q_0}(h_1, h_2, h_3)\, dt_1 dt_2 dt_3 \end{aligned}$$

Thus

$$\left|\int_{\partial C} dq \wedge dqf + 2\int_C Dqf'(q_0)\right| \leq \int\int\int_0^1 |dq \wedge dq \wedge df_{C(t)}(h_1,h_2,h_3) - dq \wedge dq \wedge df_{q_0}(h_1,h_2,h_3)|\, dt_1dt_2dt_3$$
$$< \epsilon\, |Dq(h_1,h_2,h_3)| = \epsilon \left|\int_C Dq\right|. \qquad \square$$

The corresponding characterisation of the derivative in terms of the values of the function at a finite number of points is

$$f'(q_0) = -\lim_{h_1,h_2,h_3\to 0}[Dq(h_1,h_2,h_3)^{-1}\{(h_1h_2 - h_2h_1)(f(q_0+h_3) - f(q_0)) + (h_2h_3 - h_3h_2)(f(q_0+h_1) - f(q_0)) + (h_3h_1 - h_1h_3)(f(q_0+h_2) - f(q_0))\}] \tag{3.17}$$

This is valid if it is understood that h_1, h_2, h_3 are multiples of three fixed linearly independent quaternions, $h_i = t_iH_i$, and the limit is taken as $t_1, t_2, t_3 \to 0$. The limit (3.17) is similar to that used by Joly [3, art. 54] to define ∇f for a function satisfying $\frac{\partial f}{\partial t} = 0$, which would be obtained as the Cauchy-Riemann equations if $dx \wedge dy \wedge dz$ were substituted for Dq in the definition of regularity.

To make explicit the analogy between the definition of regular quaternionic functions and the definition of regular complex functions by means of the Cauchy-Riemann equations, note that the analogues of the 2-form $\frac{1}{2}dq \wedge dq$, the 3-form $-Dq$ and the equation $d(dq \wedge dqf) = -2Dqf'(q)$ are the 0-form 1, the 1-form dz and the equation $df = dzf'(z)$. The analogy between theorem 7 and the difference-quotient definition of regular complex functions can be made explicit by stating the latter as follows: If f is regular at z_0, then given $\epsilon > 0$ there exists $\delta > 0$ such that if L is a directed line segment (i.e. an oriented parallelepiped of codimension 1 in $\mathbb{C}$) with $z_0 \in L$ and $z \in L \Rightarrow |z - z_0| < \delta$, then

$$\left|\left(\int_L dz\right)^{-1}\left(\int_{\partial L} f\right) - f'(z_0)\right| < \epsilon.$$

4 Cauchy's Theorem and the Integral Formula

The integral theorems for regular quaternionic functions have as wide a range of validity as those for regular complex functions, which is considerably wider than that of the integral theorems for harmonic functions. Cauchy's theorem holds for any rectifiable contour of integration; the integral formula, which is similar to Poisson's formula in that it gives the values of a function in the interior of a region in terms of its values on the boundary, holds for a general rectifiable boundary, and thus constitutes an explicit solution to the general Dirichlet problem.

Our route to the general form of Cauchy's theorem will be to use Goursat's method to prove the theorem for a parallelepiped, and immediately obtain the integral formula for a parallelepiped; then we can deduce that an analytic function is continuously differentiable, and use Stokes's theorem to extend Cauchy's theorem to differentiable contours. The extension to rectifiable contours also follows from an appropriate form of Stokes's theorem.

The heart of the quaternionic Cauchy's theorem is the following fact.

Theorem 8 *A differentiable function f is regular at q if and only if*

$$Dq \wedge df_q = 0.$$

Proof

$$\begin{aligned} Dq \wedge df_q(i,j,k,l) &= Dq(i,j,k)df_q(l) - Dq(j,k,l)df_q(i) \\ &\quad + Dq(k,l,i)df_q(j) - Dq(l,i,j)df_q(k) \\ &= df_q(l) + idf_q(i) + jdf_q(j) + kdf_q(k) \\ &= \Gamma_r(df_q) \end{aligned}$$

which vanishes if and only if f is regular at q, by theorem 6. □

Theorem 9 (Cauchy's theorem for a parallelepiped)
If f is regular at every point of the 4-parallelepiped C,

$$\int_{\partial C} Dq\, f = 0.$$

Proof [8] Let q_0 and $h_1, ..., h_4$ be the original vertex and edge-vectors of C. For each subset S of $\{1,2,3,4\}$ let C_S be the 4-parallelepiped with edge-vectors $\frac{1}{2}h_1, ..., \frac{1}{2}h_4$ and original vertex $q_0 + \sum_{i \in S} \frac{1}{2}h_i$; then

$$\int_{\partial C} Dqf = \sum_S \int_{\partial C_S} Dqf$$

Hence there is a C_S—call it C_1—such that

$$\left| \int_{\partial C_1} Dqf \right| \geq \frac{1}{16} \left| \int_{\partial C} Dqf \right|.$$

Now perform a similar dissection of C_1. Continuing in this way, we obtain a sequence of 4-parallelepipeds C_n, with original vertices q_n, such that C_n has edge-vectors $2^{-n}h_1, ..., 2^{-n}h_4$, $C \supset C_1 \supset C_2 \supset ...$, and

$$\left| \int_{\partial C_n} Dqf \right| \geq \frac{1}{16^n} \left| \int_{\partial C} Dqf \right|, \tag{4.1}$$

Clearly there is a point $q_\infty \in \cap C_n$, and $q_n \to q_\infty$ as $n \to \infty$.

Since f is real-differentiable at q_∞, we can write

$$f(q) = f(q_\infty) + \alpha(q - q_\infty) + (q - q_\infty)r(q)$$

where $\alpha = df_{q_\infty} \in F_1$, and $r(q) \to 0$ as $q \to q_\infty$. Then if we define $r(q_\infty) = 0$, r is a continuous function and so $|r(q)|$ has a maximum value ρ_n on ∂C_n. Since the C_n converge on q_∞, $\rho_n \to 0$ as $n \to \infty$.

Now

$$\int_{\partial C_n} Dqf(q) = \int_{C_n} d(Dq)f(q_\infty) = 0$$

and

$$\int_{\partial C_n} Dq\alpha(q-q_\infty) = \int_{C_n} d(Dq\alpha) = 16^{-n}(Dq\wedge\alpha)(h_1,...,h_4) = 0$$

by theorem 8, since f is regular at q_∞. Thus

$$\int_{\partial C_n} Dqf(q) = \int_{\partial C_n} Dq(q-q_\infty)r(q).$$

Let $F : I^3 \to \mathbb{H}$ be one of the 3-parallelepipeds forming the faces of C_n. Then $F \subset \partial C_n$, and the edge-vectors of F are three of the four edge-vectors of C_n, say $2^{-n}h_a$, $2^{-n}h_b$, and $2^{-n}h_c$. For $q \in F(I^3)$ we have $|r(q)| < \rho_n$ and $|q-q_\infty| \leq 2^{-n}(|h_1| + ... + |h_4|)$; hence

$$\left|\int_F Dq(q-q_\infty)r(q)\right| \leq 8^{-n}\,|Dq(h_a,h_b,h_c)|\,2^{-n}\,(|h_1|+...+|h_4|)\,\rho_n.$$

Let V be the largest $|Dq(h_a,h_b,h_c)|$ for all choices of a, b, c; then since the integral over ∂C_n is the sum of 8 integrals over faces F,

$$\left|\int_{\partial C_n} Dqf(q)\right| \leq 8.16^{-n}V\,(|h_1|+...+|h_4|)\,\rho_n.$$

Combining this with (4.1), we find

$$\left|\int_{\partial C} Dqf\right| \leq 8V(|h_1|+...+|h_4|)\rho_n.$$

Since $\rho_n \to 0$ as $n \to \infty$, it follows that $\int_{\partial C} Dqf = 0$. $\square$

Theorem 10 (the Cauchy-Fueter integral formula for a parallelepiped)
If f is regular at every point of the positively oriented 4-parallelepiped C, and q_0 is a point in the interior of C,

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial C}\frac{(q-q_0)^{-1}}{|q-q_0|^2}\,Dq\,f(q).$$

Proof [7] The argument of theorem 8 shows that

$$Dq\wedge df_q = -\bar{\partial}_\ell f(q)v \tag{4.2}$$

where $v = dt\wedge dx\wedge dy\wedge dz$, for any differentiable function f. A similar calculation shows that

$$df_q\wedge Dq = \bar{\partial}_r f(q)v. \tag{4.3}$$

Hence if f and g are both differentiable,

$$\begin{aligned} d(gDqf) &= d(gDq)f + gd(Dqf) \\ &= dg \wedge Dqf - gDq \wedge df \\ &= \left\{(\bar{\partial}_r g)f + g(\bar{\partial}_\ell f)\right\} v \end{aligned} \tag{4.4}$$

Take $g(q) = \frac{(q-q_0)^{-1}}{|q-q_0|^2} = \frac{\bar{q}-\bar{q}_0}{|q-q_0|^4} = \partial_r\left(\frac{1}{|q-q_0|^2}\right)$; then g is differentiable except at q_0, and

$$\bar{\partial}_r g = \Delta\left(\frac{1}{|q-q_0|^2}\right) = 0.$$

If f is regular we have $\bar{\partial}_\ell f = 0$, and so

$$d\left[\frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf\right] = 0.$$

We can now follow the argument of theorem 9 to show that

$$\int_{\partial C'} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf(q) = 0$$

where C' is any 4-parallelepiped not containing q_0. By dissecting the given 4-parallelepiped C into 81 4-parallelepipeds with edges parallel to those of C, we deduce that

$$\int_{\partial C} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf(q) = \int_{\partial C_0} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf(q),$$

where C_0 is any 4-parallelepiped containing q_0 which lies in the interior of C and has edges parallel to those of C. Take C_0 to have edge-vectors $\delta h_1, ..., \delta h_4$, where δ is a positive real number and $h_1, ..., h_4$ are the edge-vectors of C, and suppose q_0 is at the centre of C_0 (so that the original vertex of C_0 is $q_0 - \frac{1}{2}\delta h_1 - ... - \frac{1}{2}\delta h_4$); then

$$\min_{q \in \partial C_0} |q - q_0| = \min_{1 \le a,b,c \le 4} \left| \frac{v(\delta h_1, ..., \delta h_4)}{Dq(\frac{1}{2}\delta h_a, \frac{1}{2}\delta h_b, \frac{1}{2}\delta h_c)} \right| = W\delta$$

where W depends only on $h_1, ..., h_4$. Since f is continuous at q_0, we can choose δ so that $q \in C_0(I^4) \Rightarrow |f(q) - f(q_0)| < \epsilon$ for any given $\epsilon > 0$; hence

$$\left| \int_{\partial C_0} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\{f(q) - f(q_0)\} \right| \le \frac{8V}{W^3}\epsilon \tag{4.5}$$

where, as in theorem 9,

$$V = \max_{1 \le a,b,c \le 4} |Dq(h_a, h_b, h_c)|.$$

Since the 3-form $\frac{(q-q_0)^{-1}Dq}{|q-q_0|^2}$ is closed and continuously differentiable in $\mathbb{H}\backslash\{q_0\}$, Stokes's theorem gives

$$\int_{\partial C_0} \frac{(q-q_0)^{-1}\, Dq}{|q-q_0|^2} = \int_S \frac{(q-q_0)^{-1}\, Dq}{|q-q_0|^2}$$

where S is the 3-sphere $|q - q_0| = 1$, oriented so that Dq is in the direction of the outward normal to S. Working in spherical coordinates (r, θ, ϕ, ψ), in which

$$q - q_0 = r\left(\cos\theta + i\sin\theta\cos\phi + j\sin\theta\sin\phi e^{-i\psi}\right),$$

we find that on S, i.e. when $r = 1$,

$$\begin{aligned} Dq &= (q - q_0)\sin^2\theta\sin\phi d\theta \wedge d\phi \wedge d\psi \\ &= (q - q_0)dS \end{aligned} \tag{4.6}$$

where dS is the usual Euclidean volume element on a 3-sphere. Hence

$$\int_{\partial C_0} \frac{(q - q_0)^{-1}Dq}{|q - q_0|^2} = \int_S dS = 2\pi^2$$

and so (4.5) becomes

$$\left|\int_{\partial C_0} \frac{(q - q_0)^{-1}}{|q - q_0|^2}Dqf(q) - 2\pi^2 f(q_0)\right| \le \frac{8V}{W^3}\epsilon.$$

Since ϵ was arbitrary, it follows that

$$\int_{\partial C} \frac{(q - q_0)^{-1}}{|q - q_0|^2}Dqf(q) = \int_{\partial C} \frac{(q - q_0)^{-1}}{|q - q_0|^2}Dqf(q) = 2\pi^2 f(q_0). \qquad \square$$

Because of the special role played by the function occurring in this integral formula, we will use a special notation for it:

$$G(q) = \frac{q^{-1}}{|q|^2}. \tag{4.6}$$

Note that

$$G(q) = -\partial_\ell\left(\frac{1}{|q|^2}\right) = -\partial_r\left(\frac{1}{|q|^2}\right); \tag{4.7}$$

it follows that $\bar{\partial}_\ell G = 0$, i.e. G is regular except at 0.

As an immediate corollary of the integral formula we have

Theorem 11 *A function which is regular in an open set U is real-analytic in U.*

Proof Suppose f is analytic in the open set U, and let q_0 be any point of U. Then we can find a 4-parallelepiped C_0 such that q_0 lies in the interior of C_0 and C_0 lies inside U, and by theorem 10 we have

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial C_0} G(q - q_0)Dqf(q).$$

Let C_1 be a 4-parallelepiped such that $q_0 \in C_1(I^4) \subset \operatorname{int} C_0(I^4)$; then in this integral the integrand is a continuous function of (q, q_0) in $C_0(\partial I^4) \times C_1(I^4)$ and a C^∞ function of q_0 in $C_1(I^4)$. It follows that the integral defines a C^∞ function of q_0 in $C_1(I^4)$. Thus f is C^∞ throughout U. Since f is regular in U, so that $\bar{\partial}_\ell f = 0$, eq. (2.45) gives $\Delta f = 0$, i.e. f is harmonic. It follows [12, p. 269] that f is real-analytic in U. $\square$

This fact enables us to give the most general formulation of Cauchy's theorem and the integral formula for a differentiable contour of integration.

Theorem 12 (Cauchy's theorem for a differentiable contour)
Suppose f is regular in an open set U, and let C be a differentiable 3-chain in U which is homologous to 0 in the differentiable singular homology of U, i.e. $C = \partial C'$ for some differentiable 4-chain C' in U. Then

$$\int_C Dq\, f = 0.$$

Proof By theorem 8, $d(dqf) = 0$. By theorem 11, the 3-form Dqf is infinitely differentiable in U; hence we can apply Stokes's theorem to find

$$\int_C Dqf = \int_{\partial C'} Dqf = \int_{C'} d(Dqf) = 0. \qquad \square$$

In order to state the general form of the integral formula, we need an analogue of the notion of the winding number of a curve round a point in the plane.

Definition 5 *Let q be any quaternion, and let C be a closed 3-chain in $\mathbb{H}\backslash\{q\}$. Then C is homologous to a 3-chain $C' : \partial I^4 \to S$, where S is the unit sphere with centre q. The* **wrapping number** *of C about q is the degree of the map C'.*

Theorem 13 (the integral formula for a differentiable contour)
Suppose f is regular in an open set U. Let $q_0 \in U$, and let C be a differentiable 3-chain in $U\backslash\{q_0\}$ which is homologous, in the differentiable singular homology of $U\backslash\{q_0\}$, to a 3-chain whose image is ∂B for some ball $B \subset U$. Then

$$\frac{1}{2\pi^2}\int_C \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\, f(q) = nf(q_0)$$

where n is the wrapping number of C about q_0.

Proof In the case $n = 0$, C is homologous to 0 in $U\backslash\{q_0\}$, so $C = \partial C_0$ where C_0 is a differentiable 4-chain in $U\backslash\{q_0\}$. Since the 3-form $G(q-q_0)Dqf(q)$ is closed and infinitely differentiable in $U\backslash\{q_0\}$, Stokes's theorem gives

$$\frac{1}{2\pi^2}\int_C G(q-q_0)dqf(q) = \frac{1}{2\pi^2}\int_{C_0} d\left[G(q-q_0)Dqf(q)\right] = 0.$$

In the case $n = 1$, C is homologous to a 3-chain $C' : \partial I^4 \to \partial B$ where $q_0 \in B \subset U$ and the map C' has degree 1, hence C is homologous to ∂C_0, for some 4-parallelepiped C_0 with $q_0 \in \operatorname{int} C_0(I^4)$ and $C_0(I^4) \subset U$. Again using the fact that the 3-form $G(q-q_0)Dq$ is closed in $U\backslash\{q_0\}$, we have

$$\frac{1}{2\pi^2}\int_C G(q-q_0)Dqf(q) = \frac{1}{2\pi^2}\int_{\partial C_0} G(q-q_0)Dqf(q) = f(q_0)$$

by the integral formula for a parallelepiped.

For general positive n, C is homologous to a 3-chain C'' of the form $C'' = \rho \circ C'$ where C' (having image ∂B) is as in the previous paragraph and $\rho : \partial B \to \partial B$ is a map of degree n, e.g.

$$\rho\left[q_0 + r(v + jw)\right] = q_0 + r(v^n + jw)$$

where r is the radius of B. Dissect C' as $C' = \sum_{\ell=1}^{n} C'_\ell$ where the image of C'_ℓ is the sector

$$\left\{ q = q_0 + r(v + jw) \in \partial B : \frac{2\pi(\ell - 1)}{n} \leq \arg v \leq \frac{2\pi\ell}{n} \right\}.$$

Then each $\rho \circ C'_\ell$ has image ∂B and wrapping number 1 about q_0, and so by the previous paragraph

$$\frac{1}{2\pi^2} \int_C G(q - q_0) Dq f(q) = \frac{1}{2\pi^2} \sum_{\ell=1}^{n} \int_{\rho \circ C'_\ell} G(q - q_0) Dq f(q) = n f(q_0).$$

In the case $n = -1$, C is a homologous to a 3-chain C'' of the form $C'' = C' \circ K$, where $C' : \partial I^4 \to \partial B$ has degree 1 and $K : \partial I^4 \to \partial I^4$ has degree -1, for example the reflection $(t_1, t_2, t_3, t_4) \to (1 - t_1, t_2, t_3, t_4)$. Then

$$\int_C = \int_{C''} = -\int_{C'} = -2\pi^2 f(q_0),$$

the integrand $G(q - q_0) Dq f(q)$ being understood. For general negative n we dissect the 3-chain C as for general positive n. This establishes the theorem for all n. □

More generally, Cauchy's theorem and the integral formula are valid for rectifiable contours, which we define as follows.

Definition 6 *Let $C : I^3 \to \mathbb{H}$ be a continuous map of the unit 3-cube into $\mathbb{H}$, and let $P : 0 = s_0 < s_1 < \ldots < s_p = 1$, $Q : 0 = t_0 < t_1 < \ldots < t_q = 1$ and $R : 0 = u_0 < u_1 < \ldots < u_r = 1$ be three partitions of the unit interval I. Define*

$$\begin{aligned}\sigma(C; P, Q, R) = \sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \sum_{n=0}^{r-1} Dq(&C(s_{l+1}, t_m, u_n) - C(s_\ell, t_m, u_n), \\ &C(s_\ell, t_{m+1}, u_n) - C(s_\ell, t_m, u_n), \\ &C(s_\ell, t_m, u_{n+1}) - C(s_\ell, t_m, u_n)).\end{aligned}$$

C is a **rectifiable 3-cell** *if there is a real number M such that $\sigma(C; P, Q, R) < M$ for all partitions P, Q, R. If this is the case the least upper bound of the numbers $(C; P, Q, R)$ is called the* **content** *of C and denoted by $\sigma(C)$.*

Let f and g be quaternion-valued functions defined on $C(I^3)$. We say that $f Dq g$ is **integrable** *over C if the sum*

$$\begin{aligned}\sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \sum_{n=0}^{r-1} f(C(\bar{s}_\ell, \bar{t}_m, \bar{u}_n)) Dq(&C(s_{l+1}, t_m, u_n) - C(s_\ell, t_m, u_n), \\ &C(s_\ell, t_{m+1}, u_n) - C(s_\ell, t_m, u_n), \\ &C(s_\ell, t_m, u_{n+1}) - C(s_\ell, t_m, u_n)) g(C(\bar{s}, \bar{t}_m, \bar{u}_n)),\end{aligned}$$

where $s_\ell \leq \bar{s}_\ell \leq s_{\ell+1}$, $t_m \leq \bar{t}_m \leq t_{m+1}$ and $u_n \leq \bar{u}_n \leq u_{n+1}$, has a limit in the sense of Riemann-Stieltjes integration as $|P|, |Q|, |R| \to 0$, where $|P| = \max_{0 \leq \ell \leq p-1} |s_{\ell+1} - s_\ell|$ measures the coarseness of the partition P. If this limit exists, we denote it by $\int_C f Dq g$.

We extend these definitions to define rectifiable 3-chains and integrals over rectifiable 3-chains in the usual way.

Just as for rectifiable curves, we can show that $f\,Dqg$ is integrable over the 3-chain C if f and g are continuous and C is rectifiable, and

$$\left|\int_C f\,Dq\,g\right| \leq (\max_C |f|)(\max_C |g|)\sigma(C).$$

Furthermore, we have the following weak form of Stokes's theorem:

Stokes's theorem for a rectifiable contour. Let C be a rectifiable 3-chain in $\mathbb{H}$ with $\partial C = 0$, and suppose f and g are continuous functions defined in a neighbourhood U of the image of C, and that $f\,Dqg = d\omega$ where ω is a 2-form on U. Then

$$\int_C f\,Dq\,g = 0.$$

The proof proceeds by approximating C by a chain of 3-parallelepipeds with vertices at the points $C_a(s_\ell, t_m, u_n)$ where C_a is a 3-cell in C and s_ℓ, t_m, u_n are partition points in I. Stokes's theorem holds for this chain of 3-parallelepipeds, and we can use the same argument as for rectifiable curves (see e.g. [13, p. 103]).

We can now give the most general forms of Cauchy's theorem and the integral formula.

Theorem 14 (Cauchy's theorem for a rectifiable contour)
Suppose f is regular in an open set U, and let C be a rectifiable 3-chain which is homologous to 0 in the singular homology of U. Then

$$\int_C Dq\,f = 0.$$

Proof First we prove the theorem in the case when U is contractible. In this case, since $d(Dqf) = 0$ and f is continuously differentiable (by theorem 11), Poincaré's lemma applies and we have $Dqf = d\omega$ for some 2-form ω on D. But $\partial C = 0$, so by Stokes's theorem $\int_C Dqf = 0$.

In the general case, suppose $C = \partial C^*$ where C^* is a 4-chain in U. We can dissect C^* as $C^* = \sum_n C_n^*$, where each C_n^* is a 4-cell lying inside an open ball contained in U and C_n^* is rectifiable. Hence by the first part of the theorem $\int_{\partial C_n^*} Dqf = 0$, and therefore

$$\int_C Dq\,f = \sum_n \int_{\partial C_n^*} Dqf = 0. \qquad \Box$$

A similar argument proves the following general form of the integral formula

Theorem 15 (the integral formula for a rectifiable contour)
Suppose f is regular in an open set U. Let $q_0 \in U$, and let C be a rectifiable 3-chain in $U\backslash\{q_0\}$ which is homologous, in the singular homology of $U\backslash\{q_0\}$, to a differentiable 3-chain whose image is ∂B for some ball $B \subset U$. Then

$$\frac{1}{2\pi^2}\int_C \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\,f(q) = nf(q_0)$$

where n is the wrapping number of C about q_0.

5 Some Immediate Consequences

Since regular functions are harmonic, they satisfy a maximum-modulus principle and a Liouville theorem. As with functions of a complex variable, Liouville's theorem follows immediately from the Cauchy-Fueter integral formula, as in e.g. [14, p. 85 (second proof)].

Morera's theorem also holds for quaternionic functions, but in this case the usual proof cannot easily be adapted. The proof given here is based on an incomplete proof by Schuler [8].

Theorem 16 (Morera's theorem) *Suppose that the function f is continuous in an open set U and that $\int_{\partial C} Dqf = 0$ for every 4-parallelepiped C contained in U. Then f is regular in U.*

Proof The method is to show that f satisfies the integral formula and then argue as for the analyticity of a regular function (theorem 11).

First we show that $\int_{\partial C} G(q-q_0)Dqf(q) = 0$ if q_0 does not lie inside C, using Goursat's argument. As in theorem 9, we find a sequence of 4-parallelepipeds C_n, converging on a point q_∞ and satisfying

$$\left|\int_{\partial C_n} G(q-q_0)dqf(q)\right| \geq \frac{1}{16^n}\left|\int_{\partial C} G(q-q_0)dqf(q)\right|. \tag{5.1}$$

Since q_0 lies outside C, $G(q-q_0)$ is a right-regular function of q inside C and so, by the counterpart of theorem 8 for right-regular functions, $dG_{q-q_0} \wedge Dq = 0$. Now write

$$G(q-q_0) = G(q_\infty - q_0) + \alpha(q-q_\infty) + (q-q_\infty)r(q)$$

where $\alpha = dG_{q_\infty - q_0} \in F_1$, so that $\alpha \wedge Dq = 0$, and $r(q) \to 0$ as $q \to q_\infty$; and write

$$f(q) = f(q_\infty) + s(q)$$

where $s(q) \to 0$ as $q \to q_\infty$. Then

$$\begin{aligned}\int_{\partial C_n} G(q-q_0)Dqf(q) &= G(q_\infty - q_0)\int_{\partial C_n} Dqf(q) + \int_{\partial C} \alpha(q-q_\infty)Dqf(q_0) \\ &\quad + \int_{\partial C_n} \alpha(q-q_\infty)Dqs(q) + \int_{\partial C_n}(q-q_\infty)r(q)Dqf(q)\end{aligned}$$

The first term vanishes by assumption, the second because $\alpha \wedge Dq = 0$. Now for $q \in \partial C_n$ we have $|q-q_\infty| \leq 2^{-n}L$, where L is the sum of the lengths of the edges of C; since α is linear, there is a number M such that $|\alpha(q-q_\infty| \leq M|q-q_\infty|$. The volume of each face of C_n is at most $8^{-n}V$, where V is the volume of the largest face of C. Hence

$$\left|\int_{\partial C_n} G(q-q_0)Dqf(q)\right| \leq 2^{-n}LM.8^{-n}V\sigma_n + 2^{-n}L\rho_n.8^{-n}V(|f(q_\infty)| + \sigma_n)$$

where ρ_n and σ_n are the maximum values of the continuous functions $r(q)$ and $s(q)$ on ∂C_n. Since $\rho_n \to 0$ and $\sigma_n \to 0$ as $n \to \infty$, it follows that

$$16^n \left| \int_{C_n} G(q - q_0) Dq f(q) \right| \to 0 \quad \text{as} \quad n \to \infty$$

and so from (5.1) that $\int_{\partial C} G(q - q_0) Dq f(q) = 0$.

Now consider the integral $\int_{\partial C} G(q - q_0) Dq f(q)$ where q_0 lies inside C. By what we have just proved, the parallelepiped can be replaced by a small parallelepiped C_0 containing q_0, as in theorem 10. After this point has been established, the proof of theorem 10 depends only on the continuity of f and is therefore valid under the present conditions; hence

$$f(q) = \frac{1}{2\pi^2} \int_{\partial C} G(q' - q) Dq' f(q')$$

for any 4-parallelepiped C with $q \in \operatorname{int} C(I^4) \subset U$. Since $G(q' - q)$ is a continuously differentiable function of q in the interior of C as long as q' lies on its boundary, it follows, as in theorem 11, that f is differentiable and that

$$\bar{\partial}_\ell f(q) = \frac{1}{2\pi^2} \int_{\partial C} \bar{\partial}_\ell [G(q' - q)] Dq' f(q') = 0$$

since G is regular. Thus f is regular. □

6 Construction of Regular Functions

Regular functions can be constructed from harmonic functions in two ways. First, if f is harmonic then (2.45) shows that $\partial_\ell f$ is regular. Second, any real-valued harmonic function is, at least locally, the real part of a regular function:

Theorem 17 *Let u be a real-valued function defined on a star-shaped open set $U \subseteq \mathbb{H}$. If u is harmonic and has continuous second derivatives, there is a regular function f defined on U such that $\operatorname{Re} f = u$.*

Proof Without loss of generality we may assume that U contains the origin and is star-shaped with respect to it. In this case we will show that the function

$$f(q) = u(q) + 2 \operatorname{Pu} \int_0^1 s^2 \partial_\ell u(sq) q \, ds \tag{6.1}$$

is regular in U.

Since

$$\begin{aligned} \operatorname{Re} \int_0^1 s^2 \partial_\ell u(sq) q \, ds &= \frac{1}{2} \int_0^1 s^2 \left\{ t \frac{\partial u}{\partial t}(sq) + x_i \frac{\partial u}{\partial x_i}(sq) \right\} ds \\ &= \frac{1}{2} \int_0^1 s^2 \frac{d}{ds}[u(sq)] \, ds \\ &= \tfrac{1}{2} u(q) - \int_0^1 s u(sq) \, ds, \end{aligned}$$

we can write

$$f(q) = 2\int_0^1 s^2 \partial_\ell u(sq) q\, ds + 2\int_0^1 su(sq)\, ds. \tag{6.2}$$

Since u and $\partial_\ell u$ have continuous partial derivatives in U, we can differentiate under the integral sign to obtain, for $q \in U$,

$$\bar{\partial}_\ell f(q) = 2\int_0^1 s^2 \bar{\partial}_\ell \left[\partial_\ell u(sq)\right] q\, ds + \int_0^1 s^2 \left\{\partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i\right\}\, ds + 2\int_0^1 s^2 \bar{\partial}_\ell u(sq)\, ds.$$

But $\bar{\partial}_\ell \left[\partial_\ell u(sq)\right] = \frac{1}{4} s \Delta u(sq) = 0$ since u is harmonic in U, and

$$\begin{aligned} \partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i &= -2\overline{\partial_\ell u(sq)} && \text{by (2.27)} \\ &= -2\bar{\partial}_\ell u(sq) && \text{since } u \text{ is real.} \end{aligned}$$

Hence $\bar{\partial}_\ell f = 0$ in U and so f is regular. □

If the region U is star-shaped with respect not to the origin but to some other point a, formulae (6.1) and (6.2) must be adapted by changing origin, thus:

$$\begin{aligned} f(q) &= u(q) + 2\,\mathrm{Pu} \int_0^1 s^2 \partial_\ell u((1-s)a + sq)(q-a)\, ds & (6.3) \\ &= 2\int_0^1 s^2 \partial_\ell u((1-s)a + sq)(q-a)\, ds + 2\int_0^1 su((1-s)a + sq)\, ds. & (6.4) \end{aligned}$$

An example which can be expected to be important is the case of the function $u(q) = |q|^{-2}$. This is the elementary potential function in four dimensions, as $\log|z|$ is in the complex plane, and so the regular function whose real part is $|q|^{-2}$ is an analogue of the logarithm of a complex variable.

We take U to be the whole of $\mathbb{H}$ except for the origin and the negative real axis. Then U is star-shaped with respect to 1, and $|q|^{-2}$ is harmonic in U. Put

$$u(q) = \frac{1}{|q|^2}, \qquad \partial_\ell u(q) = -\frac{q^{-1}}{|q|^2}, \qquad a = 1;$$

then (6.3) gives

$$\left.\begin{aligned} f(q) &= -(q\,\mathrm{Pu}\,q)^{-1} - \frac{1}{|\,\mathrm{Pu}\,q|^2} \arg q \quad \text{if } \mathrm{Pu}\,q \neq 0 \\ &= \frac{1}{|q|^2} \quad \text{if } q \text{ is real and positive,} \end{aligned}\right\} \tag{6.5}$$

where

$$\arg q = \log(\mathrm{Un}\,q) = \frac{\mathrm{Pu}\,q}{|\,\mathrm{Pu}\,q|} \tan^{-1}\left(\frac{|\,\mathrm{Pu}\,q|}{\mathrm{Re}\,q}\right), \tag{6.6}$$

which is i times the usual argument in the complex plane generated by q. (In practice the formulae (6.3) and (6.4) are not very convenient to use, and it is easier to obtain (6.5) by solving the equations

$$\nabla.\mathbf{F} = -\frac{2t}{(t^2 + r^2)^2}$$

and

$$\frac{\partial \mathbf{F}}{\partial t} + \nabla \times \mathbf{F} = \frac{2\mathbf{r}}{(t^2+r^2)^2},$$

where $t = \operatorname{Re} q$, $\mathbf{r} = \operatorname{Pu} q$ and $r = |\mathbf{r}|$—these express the fact that $\mathbf{F} : \mathbb{H} \to P$ is the pure quaternion part of a regular function whose real part is $|q|^{-2}$—and assuming that $\mathbf{F}$ has the form $F(r)\mathbf{r}$.)

We will denote the function (6.5) by $-2L(q)$. The derivative of $L(q)$ can most easily be calculated by writing it in the form

$$L(q) = -\frac{r^2 + te_i x_i}{2r^2(r^2+t^2)} + \frac{e_i x_i}{2r^3} \tan^{-1}\left(\frac{r}{t}\right); \tag{6.7}$$

the result is

$$\partial_\ell L(q) = G(q) = \frac{q^{-1}}{|q|^2}. \tag{6.8}$$

Thus $L(q)$ is a primitive for the function occurring in the Cauchy-Fueter integral formula, just as the complex logarithm is a primitive for z^{-1}, the function occurring in Cauchy's integral formula.

Theorem 17 shows that there are as many regular functions of a quaternion variable as there are harmonic functions of four real variables. However, these functions do not include the simple algebraic functions, such as powers of the variable, which occur as analytic functions of a complex variable. Fueter [4] also found a method for constructing a regular function of a quaternion variable from an analytic function of a complex variable.

For each $q \in \mathbb{H}$, let $\eta_q : \mathbb{C} \to \mathbb{H}$ be the embedding of the complex numbers in the quaternions such that q is the image of a complex number $\zeta(q)$ lying in the upper half-plane; i.e.

$$\eta_q(x+iy) = x + \frac{\operatorname{Pu} q}{|\operatorname{Pu} q|} y, \tag{6.9}$$

$$\zeta(q) = \operatorname{Re} q + i|\operatorname{Pu} q|. \tag{6.10}$$

Then we have

Theorem 18 *Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic in the open set $U \subseteq \mathbb{C}$, and define $\tilde{f} : \mathbb{H} \to \mathbb{H}$ by*

$$\tilde{f}(q) = \eta_q \circ f \circ \zeta(q). \tag{6.11}$$

Then $\Delta\tilde{f}$ is regular in the open set $\zeta^{-1}(U) \subseteq \mathbb{H}$, and its derivative is

$$\partial_\ell(\Delta\tilde{f}) = \Delta\tilde{f}', \tag{6.12}$$

where f' is the derivative of the complex function f.

Proof [6] Writing $t = \operatorname{Re} q$, $\mathbf{r} = \operatorname{Pu} q$, $r = |\mathbf{r}|$, and $u(x,y) = \operatorname{Re} f(x+iy)$, $v(x,y) = \operatorname{Im} f(x+iy)$, the definition of $\tilde{f}$ gives

$$\tilde{f}(q) = u(t,r) + \frac{\mathbf{r}}{r} v(t,r)$$

$$\operatorname{Re}\left[\bar{\partial}_\ell \tilde{f}(q)\right] = \tfrac{1}{2}\left\{\frac{\partial}{\partial t}[u(t,r)] - \nabla.\left[\frac{\mathbf{r}}{r}v(t,r)\right]\right\} - \tfrac{1}{2}u_1(t,r) - \frac{v(t,r)}{r} - \tfrac{1}{2}v_2(t,r) \tag{6.13}$$

where the subscript 1 or 2 on u and v denotes partial differentiation with respect to the first or second variable; and

$$\begin{aligned}\mathrm{Pu}\left[\bar\partial_\ell \tilde f(q)\right] &= \tfrac12\left\{\nabla\left[u(t,r)\right] + \frac{\partial}{\partial t}\left[\frac{\mathbf r}{r}v(t,r)\right] + \nabla\times\left[\frac{\mathbf r}{r}v(t,r)\right]\right\}\\ &= \tfrac12\left\{\frac{\mathbf r}{r}u_2(t,r) + \frac{\mathbf r}{r}v_1(t,r)\right\}.\end{aligned}$$

Since $f = u+iv$ is analytic, the Cauchy-Riemann equations give $u_1 - v_2 = u_2 + v_1 = 0$ and so

$$\bar\partial_\ell \tilde f(q) = -\frac{v(t,r)}{r}.$$

Hence

$$\begin{aligned}\bar\partial_\ell \Delta \tilde f(q) = \Delta\bar\partial_\ell f(q) &= -\left(\frac{\partial^2}{\partial t^2} + \frac1r\frac{\partial^2}{\partial r^2}r\right)\frac{v(t,r)}{r}\\ &= -\frac{v_{11}(t,r)+v_{22}(t,r)}{r}\\ &= 0\end{aligned}$$

since v is a harmonic function of two variables. Thus f is regular at q if f is analytic at $\zeta(q)$.

A calculation similar to the above shows that

$$\begin{aligned}\partial_\ell \tilde f(q) &= \tfrac12(u_1+v_2) + \frac{v}{r} + \tfrac12\frac{\mathbf r}{r}(-u_2+v_1)\\ &= u_1(t,r) + \frac{\mathbf r}{r}v_1(t,r) + \frac{v(t,r)}{r}\end{aligned}$$

by the Cauchy-Riemann equations for f. But $f' = u_1 + iv_1$, so $\partial\tilde f = \tilde f' + v/r$; and $\Delta(v/r) = 0$, so

$$\partial_\ell(\Delta\tilde f) = \Delta(\partial_\ell\tilde f) = \Delta\tilde f'. \qquad \square$$

Functions of the form f have been taken as the basis of an alternative theory of functions of a quaternion variable by Cullen [15].

From (5.13) a straightforward calculation, using the fact that u and v are harmonic functions, gives

$$\Delta\tilde f(q) = \frac{2u_2(t,r)}{r} + \frac{2\mathbf r}{r}\left\{\frac{v_2(t,r)}{r} - \frac{v(t,r)}{r^2}\right\} \tag{6.14}$$

The following examples are interesting: When

$$f(z) = z^{-1}, \qquad \Delta\tilde f(q) = -4G(q); \tag{6.15}$$

when

$$f(z) = \log z, \qquad \Delta\tilde f(q) = -4L(q) \tag{6.16}$$

7 Regular Functions and Conformal Mappings

Because the quaternions are four-dimensional, very little remains in quaternionic analysis of the relation between analytic functions and conformal mappings in complex analysis. However, the conformal group of $\mathbb{H}$ acts on regular functions in a simple way. The action of rotations and inversions will be needed in studying regular polynomials, and so it seems appropriate to present here the action of the full conformal group.

Theorem 19 *Let $H^* = H \cup \{\infty\}$ be the one-point compactification of $\mathbb{H}$. If the mapping $f : H^* \to \mathbb{H}^*$ is conformal and orientation-preserving, f is of the form*

$$f(q) = (aq + b)(cq + d)^{-1} \tag{7.1}$$

for some $a, b, c, d \in \mathbb{H}$. Conversely, any such mapping is conformal and orientation-preserving.

Proof Let C be the group of orientation-preserving conformal mappings of $\mathbb{H}^*$, and let D be the set of mappings of the form (7.1). Then if $f \in D$, f has differential

$$df_q = (ac^{-1}d - b)(cq + d)^{-1}c\,dq(cq + d)^{-1}$$

This is of the form $\alpha dq\beta$, which is a combination of a dilatation and a rotation, so f is conformal and orientation-preserving. Thus D is a subset of C. Now C is generated by rotations, dilatations, translations and the inversion in the unit sphere followed by a reflection [16, p. 312], i.e. by the mappings $q \mapsto \alpha q\beta$, $q \mapsto q + \gamma$ and $q \mapsto q^{-1}$. If a mapping in D is followed by any of these mappings, it remains in D; hence $CD \subseteq D$. It follows that $D = C$. □

The same argument can be used to obtain the alternative representation $f(q) = (qc + d)^{-1}(qa + b)$.

We now show how a regular function gives rise to other regular functions by conformal transformation of the variable.

Theorem 20

(i) Given a function $f : \mathbb{H} \to \mathbb{H}$, let $If : \mathbb{H}\backslash\{0\} \to \mathbb{H}$ be the function

$$If(q) = \frac{q^{-1}}{|q|^2} f(q^{-1}).$$

If f is regular at q^{-1}, If is regular at q.

(ii) Given a function $f : \mathbb{H} \to \mathbb{H}$ and quaternions a, b, let $M(a, b)f$ be the function

$$[M(a, b)f](q) = bf(a^{-1}qb).$$

If f is regular at aqb, $M(a, b)f$ is regular at q.

(iii) Given a function $f : \mathbb{H} \to \mathbb{H}$ and a conformal mapping $\nu : q \mapsto (aq+b)(cq+d)^{-1}$, let $M(\nu)f$ be the function

$$[M(\nu)f](q) = \frac{1}{|b - ac^{-1}d|^2} \frac{(cq+d)^{-1}}{|cq+d|^2} f(\nu(q)).$$

If f is regular at $\nu(q)$, $M(\nu)f$ is regular at q.

Proof (i) By theorem 8, it is sufficient to show that

$$Dq \wedge d(If)_q = 0.$$

Now $If = G(f \circ \iota)$, where $G(q) = \frac{q^{-1}}{|q|^2}$ and $\iota : \mathbb{H}\backslash\{0\} \to \mathbb{H}$ is the inversion $q \mapsto q^{-1}$. Hence

$$\begin{aligned} Dq \wedge d(If)_q &= Dq \wedge dG_q f(q^{-1}) + Dq \wedge G(q) d(f \circ \imath)_q \\ &= Dq\, G(q) \wedge \imath_q^* df_{q^{-1}} \end{aligned}$$

since G is regular at $q \neq 0$. But

$$\begin{aligned} \imath_q^* Dq(h_1, h_2, h_3) &= Dq(-q^{-1}h_1q^{-1}, -q^{-1}h_2q^{-1}, -q^{-1}h_3q^{-1}) \\ &= -\frac{q^{-1}}{|q|^4} Dq(h_1, h_2, h_3) q^{-1} \end{aligned}$$

by (2.41). Thus

$$Dq\, G(q) = -|q|^2 q \imath_q^* Dq$$

and so

$$\begin{aligned} Dq \wedge d(If)_q &= -|q|^2 q \imath_q^* (Dq \wedge df_{q^{-1}}) \\ &= 0 \end{aligned}$$

if f is regular at q^{-1}. □

(ii) Let $\mu : \mathbb{H} \to \mathbb{H}$ be the map $q \mapsto aqb$. Then by (2.41)

$$\mu^* Dq = |a|^2 |b|^2 a\, Dq\, b$$

and so

$$\begin{aligned} Dq \wedge d[M(a,b)f]_q &= Dq \wedge b\mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2} a^{-1} (\mu_q^* Dq) b^{-1} \wedge b \mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2} a^{-1} \mu_q^* (Dq \wedge df_{\mu(q)}) \\ &= 0 \end{aligned}$$

if f is regular at $\mu(q)$. It follows from theorem 8 that $M(a,b)f$ is regular at q. □

(iii) The map $q \mapsto \nu(q) = (aq+b)(cq+d)^{-1}$ can be obtained by composing the sequence of maps

$$q \to q' = cq(b - ac^{-1}d)^{-1} \tag{7.2}$$

$$q' \to q'' = q' + d(b - ac^{-1}d)^{-1} \tag{7.3}$$

$$q'' \to q''' = q''^{-1} \tag{7.4}$$

$$q''' \to \nu(q) = q''' + ac^{-1} \tag{7.5}$$

Clearly translation preserves regularity, i.e. if f is regular at $q + \alpha$, $f(q+\alpha)$ is regular at q. Applying this to the maps (7.3), part (i) of the theorem to (7.4) and part (ii) to (7.2), we find that $M(\nu)f$ is regular at q if f is regular at $\nu(q)$. □

8 Homogeneous Regular Functions

In this section we will study the relations between regular polynomials, harmonic polynomials and harmonic analysis on the group S of unit quaternions, which is to quaternionic analysis what Fourier analysis is to complex analysis.

The basic Fourier functions $e^{in\theta}$ and $e^{-in\theta}$, regarded as functions on the unit circle in the complex plane, each have two extensions to harmonic functions on $C\backslash\{0\}$; thus we have the four functions z^n, $\bar{z}^n$, z^{-n} and $\bar{z}^{-n}$. The requirement of analyticity picks out half of these, namely z^n and z^{-n}. In the same way the basic harmonic functions on S, namely the matrix elements of unitary irreducible representations of S, each have two extensions to harmonic functions on $\mathbb{H}\backslash\{0\}$, one with a negative degree of homogeneity and one with a positive degree. We will see that the space of functions belonging to a particular unitary representation, corresponding to the space of combinations of $e^{in\theta}$ and $e^{-in\theta}$ for a particular value of n, can be decomposed into two complementary subspaces; one (like $e^{in\theta}$) gives a regular function on $\mathbb{H}\backslash\{0\}$ when multiplied by a positive power of $|q|$, the other (like $e^{-in\theta}$) has to be multiplied by a negative power of $|q|$.

Let U_n be the set of functions $f : \mathbb{H}\backslash\{0\} \to \mathbb{H}$ which are regular and homogeneous of degree n over $\mathbb{R}$, i.e.

$$f(\alpha q) = \alpha^n f(q) \qquad \text{for } \alpha \in R.$$

Removing the origin from the domain of f makes it possible to consider both positive and negative n (the alternative procedure of adding a point at infinity to $\mathbb{H}$ has disadvantages, since regular polynomials do not necessarily admit a continuous extension to $\mathbb{H} \cup \{\infty\} \cong S^4$). Let W_n be the set of functions $f : \mathbb{H}\backslash\{0\} \to \mathbb{H}$ which are harmonic and homogeneous of degree n over $\mathbb{R}$. Then U_n and W_n are right vector spaces over $\mathbb{H}$ (with pointwise addition and scalar multiplication) and since every regular function is harmonic, we have $U_n \subseteq W_n$.

Functions in U_n and W_n can be studied by means of their restriction to the unit sphere $S = \{q : |q| = 1\}$. Let

$$\tilde{U}_n = \{f|S : f \in U_n\}, \qquad \tilde{W}_n = \{f|S : f \in W_n\};$$

then U_n and $\tilde{U}_n$ are isomorphic (as quaternionic vector spaces) by virtue of the correspondence

$$f \in U_n \Leftrightarrow \tilde{f} \in \tilde{U}_n, \qquad \text{where} \qquad f(q) = r^n \tilde{f}(u), \tag{8.1}$$

using the notation $r = |q| \in \mathbb{R}$, $u = \frac{q}{|q|} \in S$.

Similarly W_n and $\tilde{W}_n$ are isomorphic.

In order to express the Cauchy-Riemann-Fueter equations in a form adapted to the polar decomposition $q = ru$, we introduce the following vector fields $X_0, ..., X_3$ on $\mathbb{H}\backslash\{0\}$:

$$X_0 f = \frac{d}{d\theta}\left[f(qe^{\theta})\right]_{\theta=0}, \tag{8.2}$$

$$X_i f = \frac{d}{d\theta}\left[f(qe^{e_i\theta})\right]_{\theta=0} = \frac{d}{d\theta} f[q(\cos\theta + e_i \sin\theta)]_{\theta=0} \quad (i = 1, 2, 3). \tag{8.3}$$

These fields form a basis for the real vector space of left-invariant vector fields on the multiplicative group of $\mathbb{H}$, and they are related to the Cartesian vector fields $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x_i}$ by

$$X_0 = t\,\frac{\partial}{\partial t} + x_i\,\frac{\partial}{\partial x_i}, \tag{8.4}$$

$$X_i = -x_i\,\frac{\partial}{\partial t} + t\,\frac{\partial}{\partial x_i} - \epsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k}, \tag{8.5}$$

$$\frac{\partial}{\partial t} = \frac{1}{r^2}(tX_0 - x_iX_i), \tag{8.6}$$

$$\frac{\partial}{\partial x_i} = \frac{1}{r^2}(\epsilon_{ijk}x_jX_k + tX_i + x_iX_0). \tag{8.7}$$

Their Lie brackets are

$$[X_0, X_i] = 0, \tag{8.8}$$

$$[X_i, X_j] = 2\epsilon_{ijk}X_k. \tag{8.9}$$

Using (3.4) and (8.5) the differential operators $\bar{\partial}_\ell$ and Δ can be calculated in terms X_0 and X_i. The result is

$$\bar{\partial}_\ell = \tfrac{1}{2}\bar{q}^{-1}(X_0 + e_iX_i), \tag{8.10}$$

$$\Delta = \frac{1}{r^2}\left\{X_iX_i + X_0(X_0+2)\right\}. \tag{8.11}$$

From (8.11) we can deduce the following (well-known) facts about W_n:

Theorem 21

(i) $\tilde{W}_n = \tilde{W}_{-n-2}$

(ii) $\dim W_n = (n+1)^2$

(iii) The elements of W_n are polynomials in q.

Proof The elements of W_n, being homogeneous of degree n, are eigenfunctions of X_0 with eigenvalue n. Since they are also harmonic, e.g. (8.11) shows that they are eigenfunctions of X_iX_i with eigenvalue $-n(n+2)$. Now the vector fields X_i are tangential to the sphere S, so their restrictions $\tilde{X}_i = X_i|S$ are vector fields on S; they are a basis for the real vector space of left-invariant vector fields on the Lie group S, which is isomorphic to SU(2). Thus if $\tilde{f} \in \tilde{W}_n$, $\tilde{f} = f|S$ for some $f \in W_n$, so f is an eigenfunction of X_iX_i with eigenvalue $-n(n+2)$ and therefore $\tilde{f}$ is an eigenfunction of $\tilde{X}_i\tilde{X}_i$ with eigenvalue $-n(n+2)$. Conversely, if $\tilde{f}$ is an eigenfunction of $\tilde{X}_i\tilde{X}_i$ with eigenvalue $-n(n+2)$, then

$$\Delta\left[r^n\tilde{f}(u)\right] = \frac{1}{r^2}\left\{r^n\tilde{X}_i\tilde{X}_i\tilde{f} + [X_0(X_0+2)r^n]\,\tilde{f}\right\} = 0$$

so $\tilde{f} \in \tilde{W}_n$. Thus $\tilde{W}_n$ is the space of eigenfunctions of $\tilde{X}_i\tilde{X}_i$ with eigenvalue $-n(n+2)$.

It follows immediately that $\tilde{W}_n = \tilde{W}_{-n-2}$. □(i)

$\tilde{W}_n$ is the quaternionification of the complex vector space $\tilde{W}_n^c$ of complex-valued functions on S which have eigenvalue $-n(n+2)$ for $\tilde{X}_i\tilde{X}_i$. It is known [17, p. 71] that this space has a basis consisting of the matrix elements of the $(n+1)$-dimensional representation of the group S. Hence

$$\dim_{\mathbb{H}} \tilde{W}_n = \dim_{\mathbb{C}} W_n^c = (n+1)^2. \qquad \square\text{(ii)}$$

In particular, $\tilde{W}_0$ consists only of constant functions, and therefore so does W_0. Now if f belongs to W_n, $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x_i}$ belong to W_{n-1}, and all the nth partial derivatives of f belong to W_0; hence all the $(n+1)$th partial derivatives of f vanish and so f is a polynomial. □(iii)

Theorem 22

(i) $\tilde{W}_n = \tilde{U}_n \oplus \tilde{U}_{-n-2}$

(ii) $U_n \cong U_{-n-3}$

(iii) $\dim U_n = \frac{1}{2}(n+1)(n+2)$

Proof (i) Eq. (8.10) shows that the elements of U_n, which satisfy $X_0 f = nf$ and $\bar{\partial}_i f = 0$, are eigenfunctions of $\Omega = e_i X_i$ with eigenvalue $-n$. As in theorem 21, it follows that U_n consists of the eigenfunctions of $\Omega = e_i X_i$ with eigenvalue $-n$. Now using (8.9), it can be shown that Ω satisfies the equation

$$\Omega^2 - 2\Omega + X_i X_i = 0$$

and therefore

$$\tilde{\Omega}^2 - 2\tilde{\Omega} + X_i X_i = 0.$$

Hence

$$\begin{aligned} \tilde{f} \in \tilde{W}_n &\Rightarrow \tilde{X}_i \tilde{X}_i f = -n(n+2)f \\ &\Rightarrow (\tilde{\Omega}_{-n-2})(\tilde{\Omega}_{+n})f = 0 \end{aligned}$$

It follows that $\tilde{W}_n$ is the direct sum of the eigenspaces of $\tilde{\Omega}$ with eigenvalues $-n$ and $n+2$ (these are vector subspaces of $\tilde{W}_n$ since the eigenvalues are real), i.e.

$$\tilde{W}_n = \tilde{U}_n \oplus \tilde{U}_{-n-2} \qquad \square\text{(i)}$$

(ii) It follows from theorem 20(i) that the mapping I is an isomorphism between U_n and U_{-n-3}. □(ii)

(iii) Let $d_n = \dim U_n$. By (1) and theorem 21 (ii),

$$d_n + d_{-n-2} = (n+1)^2$$

and by (ii), $d_{-n-2} = d_{n-1}$. Thus $d_n + d_{n-1} = (n+1)^2$. The solution of this recurrence relation, with $d_0 = 1$, is

$$d_n = \tfrac{1}{2}(n+1)(n+2). \qquad \square\text{(iii)}$$

There is a relation between theorem 20(ii) and the fact that homogeneous regular functions are eigenfunctions of Ω. Theorem 20(ii) refers to a representation M of the group $\mathbb{H}^\times \times \mathbb{H}^\times$ defined on the space of real-differentiable functions $f : \mathbb{H}\backslash\{0\} \to \mathbb{H}$ by

$$[M(a,b)f](q) = b\, f(a^{-1}qb).$$

Restricting to the subgroup $\{(a,b) : |a| = |b| = 1\}$, which is isomorphic to SU(2) $\times$ SU(2), we obtain a representation of SU(2) $\times$ SU(2). Since the map $q \mapsto aqb$ is a rotation when $|a| = |b| = 1$, the set W of harmonic functions is an invariant subspace under this representation. Now $W = \mathbb{H} \otimes_{\mathbb{C}} W^c$, where W^c is the set of complex-valued harmonic functions, and the representation of SU(2) $\times$ SU(2) can be written as

$$M(a,b)(q \otimes f) = (bq) \otimes R(a,b)f$$

where R denotes the quasi-regular representation corresponding to the action $q \mapsto aqb^{-1}$ of SU(2) $\times$ SU(2) on $\mathbb{H}\backslash\{0\}$:

$$[R(a,b)f]q = f(a^{-1}qb).$$

Thus $M|W$ is the tensor product of the representations $D^0 \times D^1$ and $R|W^c$ of SU(2) $\times$ SU(2), where D^n denotes the $(n+l)$-dimensional complex representation of SU(2). The isotypic components of $R|W^c$ are the homogeneous subspaces W_n^c, on which R acts irreducibly as $D^n \times D^n$; thus W_n is an invariant subspace under the representation M, and $M|W_n$ is the tensor product $(D^0 \times D^1) \otimes (D^n \times D^n)$. W_n therefore has two invariant subspaces, on which M acts as the irreducible representations $D^n \times D^{n+1}$ and $D^n \times D^{n-1}$. These subspaces are the eigenspaces of Ω. To see this, restrict attention to the second factor in SU(2) $\times$ SU(2); we have the representation

$$M'(b)(q \otimes f) = M(1,b)(q \otimes f) = [D^1(b)q] \otimes [R(1,b)f]$$

where $D^1(b)q = bq$. The infinitesimal generators of the representation $R(1,b)$ are the differential operators X_i; the infinitesimal generators of $D^1(b)$ are e_i (by which we mean left multiplication by e_i). Hence the infinitesimal operators of the tensor product M' are $e_i + X_i$. The isotypic components of W are the eigenspaces of the Casimir operator

$$(e_i + X_i)(e_i + X_i) = e_i e_i + X_i X_i + 2\Omega.$$

But $e_i e_i = -3$, and $X_i X_i = -n(n+2)$ on W_n; hence

$$(e_i + X_i)(e_i + X_i) = 2\Omega - n^2 - 2n - 3.$$

and so the isotypic components of W_n for the representation M' are the eigenspaces of Ω. U_n, the space of homogeneous regular functions of degree n, has eigenvalue $-n$ for Ω, and so $M'|U_n$ is the representation D^{n+1} of SU(2).

Similar considerations lead to the following fact:

Theorem 23 *If f is regular, qf is harmonic.*

Proof First we show that $f \in U_n \Rightarrow qf \in W_n$. From the definition (8.3) of the operators X_i we have

$$X_i(q) = qe_i$$

Hence

$$\begin{aligned} X_iX_i(qf) &= X_i(qe_if + qX_if) \\ &= -3qf + 2qe_iX_if + qX_iX_if \end{aligned}$$

since the X_i are real differential operators. If $f \in U_n$, it is an eigenfunction of e_iX_i with eigenvalue $-n$ and of X_iX_i with eigenvalue $-n(n+2)$, and so

$$X_iX_i(qf) = -(n+1)(n+3)f.$$

Since qf is homogeneous of degree $n+1$, it follows from (8.11) that it is harmonic. But any regular function can be represented locally as a series $f = \sum f_n$ with $f_n \in U_n$, and so the result follows for any regular f. □

The representation M of SU(2) × SU(2) can also be used to find a basis of regular polynomials. It belongs to a class of induced representations which is studied in [18], where a procedure is given for splitting the representation into irreducible components and finding a basis for each component. Rather than give a rigorous heuristic derivation by following this procedure, which is not very enlightening in this case, we will state the result and then verify that it is a basis.

Since the functions to be considered involve a number of factorials, we introduce the notation

$$\begin{aligned} z^{[n]} &= \frac{z^n}{n!} \quad \text{if } n \geq 0 \\ &= 0 \quad \text{if } n < 0 \end{aligned}$$

for a complex variable z. This notation allows the convenient formulae

$$\frac{d}{dz}z^{[n]} = z^{[n-1]}, \tag{8.12}$$

$$(z_1 + z_2)^{[n]} = \sum_r z_1^{[r]} z_2^{[n-r]}, \tag{8.13}$$

where the sum is over all integers r.

The representation D^n of $S \cong$ SU(2) acts on the space of homogeneous polynomials of degree n in two complex variables by

$$[D^n(u)f](z_1, z_2) = f(z_1', z_2'),$$

where

$$z_1' + jz_2' = u^{-1}(z_1 + jz_2).$$

Writing $u = v + jw$ where $v, w \in \mathbb{C}$ and $|v|^2 + |w|^2 = 1$, we have

$$z_1' = \bar{v}z_1 + \bar{w}z_2, \qquad z_2' = -wz_1 + vz_2.$$

Hence the matrix elements of $D^n(u)$ relative to the basis $f_k(z_1, z_2) = z_1^{[k]} z_2^{[n-k]}$ are

$$D^n_{k\ell}(u) = (-)^n k!(n-k)! P^n_{k\ell}(u)$$

where

$$P^n_{k\ell}(v+jw) = \sum_r (-)^r v^{[n-k-\ell+r]}\bar{v}^{[r]} w^{[k-r]}\bar{w}^{[\ell-r]} \tag{8.14}$$

The functions $P^n_{k\ell}(q)$ are defined for all quaternions $q = v + jw$ and for all integers k, ℓ, n, but they are identically zero unless $0 \le k, \ell \le n$.

Theorem 24 *As a right vector space over* $\mathbb{H}$, U_n *has the basis*

$$Q^n_{k\ell}(q) = P^n_{k\ell}(q) - jP^n_{k-1,\ell}(q) \qquad (0 \le k \le \ell \le n).$$

Proof Using (8.14), it is easy to verify that $Q^n_{k\ell}$ satisfies the Cauchy-Riemann-Fueter equations in the form

$$\frac{\partial P^n_{k\ell}}{\partial \bar{v}} = -\frac{\partial P^n_{k-1,\ell}}{\partial \bar{w}}, \qquad \frac{\partial P^n_{k\ell}}{\partial w} = \frac{\partial P^n_{k-1,\ell}}{\partial v}.$$

(cf. 3.16). Since the functions $D^n_{k\ell}$ are independent over $\mathbb{C}$ as functions on S for $0 \le k$, $\ell \le n$, the functions $P^n_{k\ell}$ are independent over $\mathbb{C}$ as functions on $\mathbb{H}$ for $0 \le k$, $\ell \le n$. It follows that the functions $Q^n_{k\ell}$ ($0 \le k \le n+1$, $0 \le \ell \le n$) are independent over $\mathbb{C}$ and therefore span a right vector space over $\mathbb{H}$ of dimension at least $\frac{1}{2}(n+1)(n+2)$. Since this space is a subspace of U_n, which has dimension $\frac{1}{2}(n+1)(n+2)$, the $Q^n_{k\ell}$ span U_n.

Since $zj = j\bar{z}$ for any $z \in \mathbb{C}$, it can be seen from the definition (8.14) that

$$P^n_{k\ell}\, j = j\, P^n_{n-k,n-\ell}$$

and therefore

$$Q^n_{k\ell}\, j = Q^n_{n-k+1,n-\ell}.$$

Thus U_n is spanned by the $Q^n_{k\ell}$ ($0 \le k \le \ell \le n$), which therefore form a basis for U_n. $\square$

Another basis for U_n will be given in the next section.

We conclude this section by studying the quaternionic derivative ∂_ℓ. Since ∂_ℓ is a linear map from U_n into U_{n-1} and $\dim U_n > \dim U_{n-1}$, ∂_ℓ must have a large kernel and so we cannot conclude from $\partial_\ell f = 0$ that f is constant. However, although the result is far from unique, it is possible to integrate regular polynomials:

Theorem 25 *Every regular polynomial has a primitive, i.e.* ∂_ℓ *maps* U_n *onto* U_{n-1} *if* $n > 0$.

Proof Suppose $f \in U_n$ is such that $\partial_\ell f = 0$. Then

$$\frac{\partial f}{\partial t} = e_i \frac{\partial f}{\partial x_i} = 0.$$

Thus f can be regarded as a function on the space P of pure imaginary quaternions. Using vector notation for elements of P and writing $f = f_0 + \mathbf{f}$ with $f_0 \in R$, $\mathbf{f} \in P$, the condition $e_i \frac{\partial f}{\partial x_i} = 0$ becomes

$$\nabla f_0 + \nabla \times \mathbf{f} = 0, \qquad \nabla . \mathbf{f} = 0.$$

If $n \geq 0$, we can define $f(0)$ so that these hold throughout P, and so there exists a function $\mathbf{F} : P \to P$ such that

$$\mathbf{f} = \nabla \times \mathbf{F}, \qquad f_0 = -\nabla . \mathbf{F},$$

i.e.

$$f = e_i \frac{\partial \mathbf{F}}{\partial x_i}.$$

Then $\mathbf{F}$ is harmonic, i.e. $\nabla^2 \mathbf{F} = 0$.

Let T_n be the right quaternionic vector space of functions $\mathbf{F} : P \to \mathbb{H}$ which are homogeneous of degree n and satisfy $\nabla^2 \mathbf{F} = 0$; then $\dim T_n = 2n+1$. Let K_n be the subspace of T_n consisting of functions satisfying $e_i \frac{\partial \mathbf{F}}{\partial x_i} = 0$; then $K_n = \ker \partial_\ell \subset U_n$. The above shows that $e_i \frac{\partial}{\partial x_i} : T_{n+1} \to T_n$ maps T_{n+1} onto K_n; its kernel is K_{n+1}, and so

$$\dim K_n + \dim K_{n+1} = \dim T_{n+1} = 2n+3.$$

The solution of this recurrence relation, with $\dim K_0 = 1$, is $\dim K_n = n+1$. But

$$\dim U_n - \dim U_{n-1} = \tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}n(n+1) = n+1.$$

It follows that ∂_ℓ maps U_n onto U_{n-1}. $\square$

Theorem 26 *If $n < 0$, the map $\partial_\ell : U_n \to U_{n-1}$ is one-to-one.*

Proof We introduce the following inner product between functions defined on the unit sphere S:

$$\langle f, g \rangle = \int_S \overline{f(u)} g(u) du$$

where du denotes Haar measure on the group S, normalised so that $\int_S du = \frac{1}{2}\pi^2$. For functions defined on $\mathbb{H}$, we can use (4.6) to write this as

$$\langle f, g \rangle = \int_S \overline{f(q)} q^{-1} Dq \, g(q)$$

As a map : $U_n \times U_n \to \mathbb{H}$, this is antilinear in the first variable and linear in the second, i.e.

$$\langle fa, gb \rangle = \bar{a} \langle f, g \rangle b \quad \text{for all } a, b \in \mathbb{H}$$

and is non-degenerate since $\langle f, f \rangle = 0 \iff f = 0$.

Now let $f \in U_n$, $g \in U_{-n-2}$ and let I denote the map : $U_n \to U_{-n-3}$ defined in theorem 20(i). Then

$$\begin{aligned} \langle g, I \partial_\ell f \rangle &= \int_S \overline{g(q)} q^{-1} Dq \, q^{-1} \partial_\ell f(q^{-1}) \\ &= -\int_S \overline{g(q)} \imath^* (Dq \, \partial_\ell f), \end{aligned}$$

where $\imath$ denotes the map $q \mapsto q^{-1}$ and we have used the fact that $\imath^* Dq = -q^{-1}Dqq^{-1}$ for $q \in S$. Since f is regular, $Dq\partial_\ell f = \frac{1}{2}d(dq \wedge dqf)$ and so

$$\begin{aligned}\langle g, I\partial_\ell f\rangle &= -\frac{1}{2}\int_S \overline{g(q)}d[\imath^*(dq \wedge dq\, f)] \\ &= -\frac{1}{2}\int_S d\bar{g} \wedge \imath^*(dq \wedge dq\, f) \quad \text{since } \partial S = 0.\end{aligned}$$

On S, the inversion $\imath$ coincides with quaternion conjugation; hence $\imath^* dq = d\bar{q}$ and therefore

$$\begin{aligned}\langle g, I\partial_\ell f\rangle &= -\frac{1}{2}\int_S d\bar{g} \wedge d\bar{q} \wedge d\bar{q} f(q^{-1}) \\ &= -\frac{1}{2}\int_S \overline{dq \wedge dq \wedge dg} f(q^{-1}) \\ &= -\int_S \overline{Dq \wedge \partial_\ell g(q)} f(q^{-1})\end{aligned}$$

since g is regular. Since conjugation is an orthogonal transformation with determinant -1, $Dq(\bar{h}_1, \bar{h}_2, \bar{h}_3) = -\overline{Dq(h_1, h_2, h_3)}$; hence, because conjugation is the same as inversion on S,

$$\overline{Dq} = -\imath^* Dq = q^{-1}Dq\, q^{-1}.$$

Thus

$$\begin{aligned}\langle g, I\partial_\ell f\rangle &= -\int_S \overline{\partial_\ell g(q)} q^{-1} Dq\, q^{-1} f(q^{-1}) \\ &= -\langle \partial_\ell g, If\rangle.\end{aligned}$$

But I is an isomorphism, the inner product is non-degenerate on U_{-n-2}, and ∂_ℓ maps U_{-n-2} onto U_{-n-3} if $n \leq -3$; it follows that $\partial_\ell : U_n \to U_{n-1}$ is one-to-one. □

In the missing cases $n = -1$ and $n = -2$, theorems 25 and 26 are both true trivially, since $U_{-1} = U_{-2} = \{0\}$.

9 Regular Power Series

The power series representing a regular function, and the Laurent series representing a function with an isolated singularity, are most naturally expressed in terms of certain special homogeneous functions.

Let ν be an unordered set of n integers $\{i_1, ..., i_n\}$ with $1 \leq i_r \leq 3$; ν can also be specified by three integers n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$; where n_1 is the number of 1's in ν, n_2 the number of 2's and n_3 the number of 3's, and we will write $\nu = [n_1 n_2 n_3]$. There are $\frac{1}{2}(n+1)(n+2)$ such sets ν; we will denote the set of all of them by σ_n. They are to be used as labels; when $n = 0$, so that $\nu = \emptyset$, we use the suffix 0 instead of $\emptyset$. We write ∂_ν for the n'th order differential operator

$$\partial_\nu = \frac{\partial^n}{\partial x_{i_1} ... \partial x_{i_n}} = \frac{\partial^n}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}}.$$

The functions in question are

$$G_\nu(q) = \partial_\nu G(q) \tag{9.1}$$

and

$$P_\nu(q) = \frac{1}{n!}\sum (te_{i_1} - x_{i_1})...(te_{i_n} - x_{i_n}) \tag{9.2}$$

where the sum is over all $\frac{n!}{n_1!n_2!n_3!}$ different orderings of n_1 1's, n_2 2's and n_3 3's. Then P_ν is homogeneous of degree n and G_ν is homogeneous of degree $-n-3$.

As in the previous section, U_n will denote the right quaternionic vector space of homogeneous regular functions of degree n.

Theorem 27 *The polynomials P_ν ($\nu \in \sigma_n$) are regular and form a basis for U_n.*

Proof [17] Let f be a regular homogeneous polynomial of degree n. Since f is regular

$$\frac{\partial f}{\partial t} + \sum_i e_i \frac{\partial f}{\partial x_i} = 0$$

and since it is homogeneous,

$$t\frac{\partial f}{\partial t} + \sum_i x_i \frac{\partial f}{\partial x_i} = 0.$$

Hence

$$nf(q) = \sum_i (x_i - te_i)\frac{\partial f}{\partial x_i}.$$

But $\frac{\partial f}{\partial x_i}$ is regular and homogeneous of degree $n-1$, so we can repeat the argument; after n steps we obtain

$$\begin{aligned} f(q) &= \frac{1}{n!}\sum_{i_1...i_n}(x_{i_1} - te_{i_1})...(x_{i_n} - te_{i_n})\frac{\partial^n f}{\partial x_{i_1}...\partial x_{i_n}} \\ &= \sum_{\nu\in\sigma_n}(-1)^n P_\nu(q)\partial_\nu f(q). \end{aligned}$$

Since f is a polynomial, $\partial_\nu f$ is a constant; thus any regular homogeneous polynomial is a linear combination of the P_ν. Let V_n be the right vector space spanned by the P_ν. By theorem 21(iii), the elements of U_n are polynomials, so $U_n \subseteq V_n$; but $\dim V_n \leq \frac{1}{2}(n+1)(n+2) = \dim U_n$ by theorem 22(iii). Hence $V_n = U_n$. □

The mirror image of this argument proves that the P_ν are also right-regular.

Theorem 28 *The expansions*

$$\begin{aligned} G(p-q) &= \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} P_\nu(q)G_\nu(p) & (9.3) \\ &= \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} G_\nu(p)P_\nu(q) & (9.4) \end{aligned}$$

are valid for $|q| < |p|$; the series converge uniformly in any region $\{(p,q) : |q| \leq r|p|\}$ with $r < 1$.

Proof Just as for a complex variable, we have

$$(1-q)^{-1} = \sum_{n=0}^{\infty} q^n$$

for $|q| < 1$; the series converges absolutely and uniformly in any ball $|q| \le r$ with $r < 1$. Hence $G(1-q) = (1-q)^{-2}(1-\bar{q})$ can be expanded as a power series in q and $\bar{q}$ which converges uniformly in any ball with radius less than 1. Because G has the multiplicative property

$$G(q_1 q_2) = G(q_2)G(q_1), \tag{9.5}$$

it follows that $G(p-q)$ can be expanded as a power series in $p^{-1}q$, multiplied by $G(p)$; the series converges uniformly in any region $|p^{-1}q| \le r$ with $r < 1$.

Regarding this series as a power series in q and identifying it with the Taylor series of G about p, we have

$$G(p-q) = \sum_{r,s=0}^{\infty} \sum_{i_1 \dots i_s} \frac{(-1)^{r+s}}{(r+s)!} \frac{\partial^{r+s} G}{\partial t^r \partial x_{i_1} \dots \partial x_{i_s}}(p)\, t^r x_{i_1} \dots x_{i_s}. \tag{9.6}$$

Since G is regular, each derivative with respect to t can be replaced by the combination $-\sum_i e_i \frac{\partial}{\partial x_i}$, giving

$$\begin{aligned} G(p-q) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r+s=n} \sum_{\substack{j_1 \dots j_r \\ i_1 \dots i_s}} (te_{j_1}) \dots (te_{j_r})(-x_{i_1}) \dots (-x_{i_s}) \frac{\partial^n G}{\partial x_{j_1} \dots \partial x_{j_r} \partial x_{i_1} \dots \partial x_{i_s}}(p) \\ &= \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q) G_\nu(p). \end{aligned}$$

This proves (9.5). But G is also right-regular, i.e.

$$\frac{\partial G}{\partial t} = -\sum_i \frac{\partial G}{\partial x_i} e_i,$$

so the derivatives with respect to t in (9.6) can alternatively be replaced by combinations of derivatives with respect to the x_i with coefficients on the right, thus giving

$$G(p-q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} G_\nu(p) P_\nu(q). \qquad \square$$

Theorem 29 *Suppose f is regular in a neighbourhood of 0. Then there is a ball B with centre 0 in which $f(q)$ is represented by a uniformly convergent series*

$$f(q) = \sum_{n=0} \sum_{\nu \in \sigma_n} P_\nu(q) a_\nu, \tag{9.7}$$

where the coefficients a_ν are given by

$$\begin{aligned} a_\nu &= \frac{1}{2\pi^2}\int_{\partial B} G_\nu(q)\, Dq\, f(q) && (9.8)\\ &= (-1)^n \partial_\nu f(0). && (9.9)\end{aligned}$$

Proof Let S be a sphere with centre 0 lying inside the domain of regularity of f, B a closed ball with centre 0 lying inside S. Then for $q \in B$ the integral formula gives

$$\begin{aligned} f(q) &= \frac{1}{2\pi^2}\int_S G(q'-q)Dq' f(q') \\ &= \frac{1}{2\pi^2}\int_S \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} P_\nu(q)G_\nu(q')Dq' f(q') \end{aligned}$$

since $|q| < |q'|$ if $q \in B$ and $q' \in S$. The series converges uniformly on $B \times S$, so it can be integrated term by term to give a uniformly convergent series of the form (9.7) with

$$a_\nu = \frac{1}{2\pi^2}\int_S G_\nu(q)Dqf(q).$$

But the functions G_ν are right-regular except at 0, so (4.4) gives $d(G_\nu Dqf) = 0$ outside B, and therefore Stokes's theorem can be used to replace the contour S by ∂B.

Differentiating the integral formula gives

$$\partial_\nu f(q) = \frac{(-1)^n}{2\pi^2}\int_S G_\nu(q'-q)Dq' f(q')$$

Hence $\partial_\nu f(0) = (-1)^n a_\nu$. □

Corollary

$$\frac{1}{2\pi^2}\int_S G_\mu(q)\, Dq\, P_\nu(q) = \delta_{\mu\nu},$$

where S is any sphere containing the origin.

This follows by putting $f = P_\mu$ in (9.7).

Theorem 30 (the Laurent series) *Suppose f is regular in an open set U except possibly at $q_0 \in U$. Then there is a neighbourhood N of q_0 such that if $q \in N$ and $q \neq q_0$, $f(q)$ can be represented by a series*

$$f(q) = \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n}\{P_\nu(q-q_0)a_\nu + G_\nu(q-q_0)b_\nu\}$$

which converges uniformly in any hollow ball

$$\{q : r \le |q-q_0| \le R\}, \quad \text{with } r > 0\text{, which lies inside } N.$$

The coefficients a_ν and b_ν are given by

$$\begin{aligned} a_\nu &= \frac{1}{2\pi^2}\int_C G_\nu(q-q_0)\, Dq\, f(q), && (9.11)\\ b_\nu &= \frac{1}{2\pi^2}\int_C P_\nu(q-q_0)\, Dq\, f(q), && (9.12)\end{aligned}$$

where C is any closed 3-chain in $U\backslash\{q_0\}$ which is homologous to ∂B for some ball B with $q_0 \in B \subset U$ (so that C has wrapping number 1 about q_0).

Proof Choose R_1 so that the closed ball $B_1 = \{q : |q - q_0| \leq R_1\}$ lies inside U, and let $N = \operatorname{int} B_1$, $S_1 = \partial B_1$. Given $q \in N\backslash\{q_0\}$, choose R_2 so that $0 < R_2 < |q - q_0| < R_1$, and let S_2 be the sphere $\{q : |q - q_0| = R_2\}$. Then by the integral formula,

$$f(q) = \frac{1}{2\pi^2}\int_{S_1} G(q' - q)Dq' f(q') - \frac{1}{2\pi^2}\int_{S_2} G(q' - q)Dq' f(q'). \qquad (9.13)$$

For $q' \in S_1$ we have $|q' - q_0| < |q - q_0|$, and so as in theorem 29

$$\frac{1}{2\pi^2}\int_{S_1} G(q' - q)Dq' f(q') = \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} P_\nu(q - q_0)a_\nu \qquad (9.14)$$

where

$$a_\nu = \frac{1}{2\pi^2}\int_{S_1} G_\nu(q' - q_0)Dq' f(q')$$

and the series is uniformly convergent in any ball $|q-q_0| \leq R$ with $R < R_1$. If C is a 3-chain as in the statement of the theorem, it is homologous to S_1; since $d(G_\nu Dqf) = 0$ in $U\backslash\{q_0\}$, Stokes's theorem gives (9.11).

For $q' \in S_2$ we have $|q' - q_0| < |q - q_0|$ and so we can expand $G(q' - q)$ as in (9.4) to obtain

$$\begin{aligned}-\frac{1}{2\pi^2}\int_{S_2} G(q' - q)Dq' f(q') &= \frac{1}{2\pi^2}\int_{S_2}\sum_{n=0}\sum_{\nu\in\sigma_n} G_\nu(q - q_0)P_\nu(q' - q_0)Dq' f(q')\\ &= \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} G_\nu(q - q_0)b_\nu\end{aligned} \qquad (9.15)$$

where

$$b_\nu = \frac{1}{2\pi^2}\int_{S_2} P_\nu(q' - q_0)Dq' f(q').$$

The series is uniformly convergent in any region $|q - q_0| \geq r$ with $r > R_2$. Since the P_ν are right-regular, $d(P_\nu Dqf) = 0$ in $U\backslash\{q_0\}$ and so the contour S_2 in the formula for b_ν can be replaced by any 3-chain C as in the statement of the theorem. In particular, b_ν is independent of the choice of R_2 and so the same representation is valid for all $q \in N\backslash\{q_0\}$. Now putting (9.14) and (9.15) into (9.13) gives (9.10) for all $q \in N\backslash\{q_0\}$. □

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